Generalized scaling and integrability from $AdS_5 \times S^5$

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Outline

- Introduction
- AdS/CFT in a nutshell
- What are the cusp anomaly and generalized scaling function?
- Superstring in the light cone gauge
- Extracting scaling function from string theory and comparison to gauge theory

- Understand quantum gauge theories at any coupling.
- Make best use of symmetries by choosing simplest non trivial gauge theory.

Ideal candidate: $\mathcal{N} = 4$ SYM

- It has maximal symmetry
- It is conformal
- Infinite dimensional symmetry: Integrability
- It is dual to a superstring theory in a non-trivial background

AdS/CFT

- By the AdS/CFT correspondence, this gauge theory is believed to be dual to type IIB string theory on $AdS_5 \times S^5$.

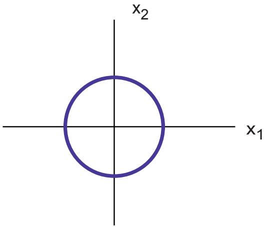
$$\alpha' \text{ expansion} \leftrightarrow \frac{1}{\sqrt{\lambda}} \qquad \text{genus expansion} \leftrightarrow \frac{1}{N} \text{ expansion}$$

- This is a strong/weak duality which is in general very hard to test directly.

Can we bridge the gap between weak and strong coupling regimes?

...sometime we have "interpolating functions".

Simplest example: Half-BPS circular Wilson loop



$$W \sim \mathrm{Tr} P \exp \left(\oint A + \Phi \right)$$

The VEV is *non trivial* and it is exactly computable for any λ!

$$\langle W \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \, , \qquad N \to \infty$$

See also Bassetto's talk; Supersymmetry is the key.

Non-supersymmetric example: cusp anomaly

No simple formula but an integral equation exists.

Beisert Eden Staudacher

A perturbative solution can be extracted at any desired order.

The cusp anomaly is almost ubiquitous:

- Renormalization of a light-like Wilson loops with cusps
- Gluon scattering amplitudes
- It governs the logarithmic scaling of high spin “twist” operators.

The simplest case is "twist two" (i.e. only two $\mathcal{Z}$ fields)

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{Z} D_+^S \mathcal{Z}\right) \,, \qquad \mathcal{Z} = \Phi^1 + i\Phi^2$$

This is the $\mathcal{N} = 4$ analogue of a QCD operator like

$$\bar{q}\, \gamma_+ D_+^S\, q \,, \qquad q = \text{quark}$$

The conformal dimension can be read from the 2-point correlator

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{1}{(x-y)^{2\Delta(S)}}$$

$$\Delta(S) = 2 + S + \delta(S)$$

For large spin we have a logarithmic scaling

$$\delta(S) = f(\lambda) \log S \,, \qquad S \to \infty$$

Comparing the spectra on the two sides of the duality:

- Gauge theory side: Compute the conformal dimension of local operators of *planar* $(N \to \infty)$ $\mathcal{N} = 4$ SYM
- AdS side: Compute the energy of *free* string in $AdS_5 \times S^5$

According to AdS/CFT duality it is the same computation

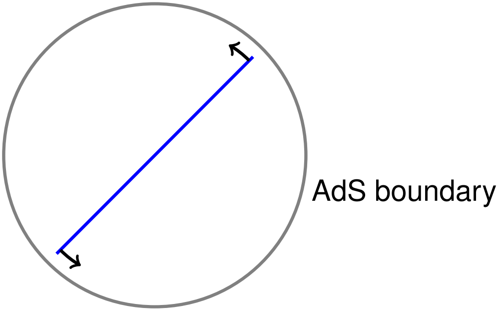
$$\text{Conformal dimension} \ \ \Delta(\lambda) = \text{String energy} \ \ \mathrm{E}(\alpha')$$

What is the string in $AdS_5 \times S^5$ dual to the twist operator?

Folded spinning string

It is a spinning closed string ("folded" on itself) in AdS_3

Gubser Klebanov Polyakov



It is pointlike in S^5

Why this is the correct string dual?

$$E_{\text{classical}} = S + \frac{\sqrt{\lambda}}{\pi} \log S\,,$$

This logarithmic behaviour persists after including quantum corrections

$$E_{\text{one loop}} = -\frac{3\log 2}{\pi} \log S \qquad \text{Frolov Tseytlin}$$

These results combined with the behaviour at weak coupling

$$\Delta = S + (\alpha_1\lambda + \alpha_2\lambda^2 + \cdots) \log S$$

support the idea that we have interpolation between weak and strong coupling

$$\Delta = E_{\text{string}} = S + f(\lambda) \log S$$

Generalized scaling

A new interpolating function appears when we additionally turn on one (large) angular momentum J in S^5.

This corresponds in gauge theory to the operator

$$\mathcal{O} \sim \mathrm{Tr}(D_+^S \mathcal{Z}^J)$$

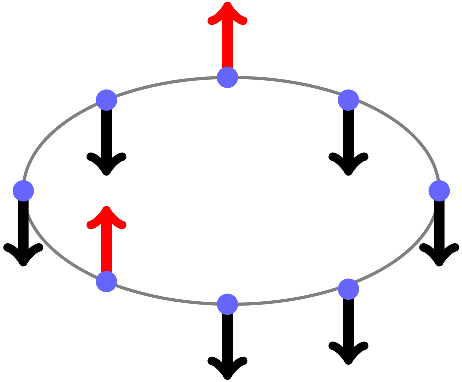
The conformal dimension is a non-trivial function

$$\Delta(S, J, \sqrt{\lambda})$$

It can be explored in various regimes of the parameters testing important features of gauge/string duality.

Interpolation between BMN-like (*large* J, *small* S) and minimal twist ($J = 2$) operators.

Integrability allows to map the one-loop anomalous dimension into the energy of an SL(2) Heisenberg spin-chain.



$$\mathrm{Tr}\,(\mathcal{Z}\ldots\mathcal{Z}\ldots\mathcal{Z}) \equiv |\downarrow\;\ldots\downarrow\ldots\downarrow\rangle\,, \qquad \text{spin chain vacuum}$$

$$\mathrm{Tr}\left(\mathcal{Z}\ldots\; D_+\mathcal{Z}\;\ldots\mathcal{Z}\right) \equiv |\downarrow\;\ldots\uparrow\ldots\downarrow\rangle \qquad \text{excitation (magnon)}$$

The Bethe equations determine the allowed momenta for the S magnons. At one-loop ($u_k \sim \cot(p_k/2)$):

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^J = \prod_{j \neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}, \qquad \prod_{k=1}^{S} \frac{u_k + i/2}{u_k - i/2} = 1$$

$$\delta(J, S) = g^2 \sum_{k=1}^{S} \frac{2}{u_k^2 + 1/4}$$

An all loop generalization for the Bethe equations exists

Beisert Eden Staudacher

- Solve them at strong coupling and compare with string prediction
- Equations simplify in the semiclassical scaling limit

$$\lambda \gg 1 \qquad \frac{S}{\sqrt{\lambda}} \gg 1 \qquad \frac{J}{\sqrt{\lambda}} \gg 1 \qquad \ell \equiv \frac{\pi J}{\sqrt{\lambda}\log S} = \text{fixed}$$

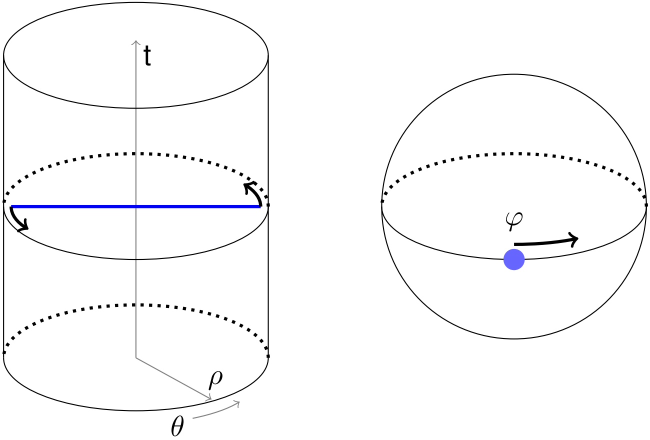
The anomalous dimension scales logarithmically in the spin

$$\delta(\ell, S) = \frac{\sqrt{\lambda}}{\pi} \mathrm{f}(\ell, \lambda) \log S$$

$\mathrm{f}(\ell, \lambda)$ is the generalized scaling dimension: new interpolating function

Belitsky Gorsky Korchemsky; Freyhult, Rej, Staudacher

The string is a folded segment passing through AdS center up to the boundary and rotating in S^1 inside S^5



Length of the string is controlled by the spin: $\log S$

Classically

$$\mathrm{f}_0(\ell) = \sqrt{1+\ell^2}$$

Quantum corrections $\sim 1/(\sqrt{\lambda})^n$ modify the classical result

$$\mathrm{f}(\ell,\lambda) = \mathrm{f}_0(\ell) + \frac{1}{\sqrt{\lambda}}\mathrm{f}_1(\ell) + \frac{1}{\lambda}\mathrm{f}_2(\ell) + \cdots$$

Up to one-loop string theory and Bethe-Ansatz agree

Frolov-Tirziu-Tseytlin; Casteill-Kristjansen; Belitsky

Two-loops string theory computation

$$\mathrm{f}_2(\ell,\lambda) = -\mathrm{K} + \ell^2 \left(8 \log^2 \ell - 6 \log \ell + q_0 \right) + \mathcal{O}(\ell^4)$$

$$q_0 = -\frac{3}{2} \log 2 + \frac{7}{4} - 2\mathrm{K} \,, \qquad \text{String (?)}$$

Roiban-Tseytlin '07

Almost equal to the Bethe-Ansatz prediction

$$q_0 = -\frac{3}{2} \log 2 + \frac{11}{4} \,, \qquad \text{Bethe Ansatz}$$

Gromov '08

$$\mathrm{K} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sim 0.9159...$$

Status so far:

- We compared the results at strong coupling for the conformal dimension of the operator

$$\mathcal{O} \sim \mathrm{Tr}(D^S \Phi^J)$$

in gauge theory (Bethe-Ansatz) and in string theory.

- Disagreement emerges at 2-loops...

Breakdown of integrability in string theory??

- We will perform the string computation again, but this time using a different "*gauge*".

A thermodynamical analogy

We need to compute $E - S$ and J for our semiclassical string.

Suppose we have a $2d$ σ-model with conserved charges Q_i

$$\tilde{H}_{2d} = H_{2d} + \sum_i \mu_i Q_i \,, \qquad Z(h_i) = e^{-\beta\Sigma(\mu_i)} = \mathrm{Tr} e^{-\beta\tilde{H}}$$

Averages $\langle Q_i \rangle$ can be computed differentiating w.r.t. μ_i. In our context it is natural to consider

$$\tilde{H}_{2d} = H_{2d} + \kappa(E - S) - \nu J$$

The parameters κ and ν explicitly appear in the string solution. They are *not* independent

$$\tilde{H}_{2d} = 0 \to \kappa = \kappa(\nu) \qquad\qquad \text{Virasoro}$$

$$-\log Z_{\text{grand.can.}} = \langle U \rangle - \frac{1}{\beta}\langle \mathcal{S} \rangle - \mu \langle N \rangle$$

For an infinite worldsheet ($\beta \sim \log S \to \infty$)

$$\Sigma(\nu) = \langle H_{2\text{d}} \rangle + \kappa \langle E - S \rangle - \nu \langle J \rangle \,, \qquad \frac{d\Sigma(\nu)}{d\nu} = \frac{d\kappa(\nu)}{d\nu}\langle E - S \rangle - \langle J \rangle$$

We can solve these two equations to extract $\langle E - S \rangle$ and $\langle J \rangle$.

We obtain

$$\langle E-S\rangle = \sqrt{1+\nu^2}\left(\Sigma(\nu) - \nu\frac{d\Sigma(\nu)}{d\nu}\right) ,$$

Remember

$$E-S = \frac{\sqrt{\lambda}}{\pi}\mathrm{f}(\ell,\lambda)\log S$$

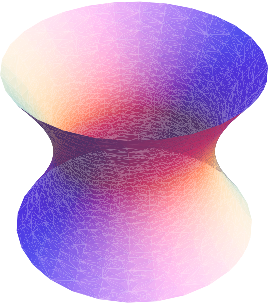
$\langle E-S\rangle$ scales logarithmically because the effective action Σ is proportional to the worldsheet volume:

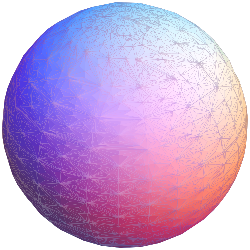
$$\Sigma \propto \mathrm{Vol} \propto \log S$$

We can therefore read the generalized scaling

$$\mathrm{f}(\ell,\lambda) = \sqrt{1+\nu^2}\left(\mathcal{F}(\nu) - \nu\frac{d\mathcal{F}(\nu)}{d\nu}\right), \qquad \mathcal{F} \sim \Sigma/\mathrm{Vol}$$

$\mathrm{AdS}_5 \times S^5$ superstring





It is based on the supercoset $\frac{G}{H} = \frac{PSU(2,2|4)}{SO(2,3)SO(5)} \supset AdS_5 \times S^5$

Metsaev, Tseytlin

$$S \sim \sqrt{\lambda} \int G_{MN} \partial X^M \partial X^N + \bar{\theta}(D + F_5)\theta \partial X + \cdots$$

- Classically integrable

  Lüscher, Pohlmeyer; Bena, Polchinski, Roiban

- Infinite tower of conserved charges:
  local (Noether)+non-local charges

Performing a conformal transformation on the folded string we can do all computations in the Poincare patch which makes life easier

$$ds^2_{AdS_5} + ds^2_{S^5} = \frac{dx^a dx^a + dz^M dz^M}{z^2}$$

We still need to fix the worldsheet symmetries.

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Light-cone

Superstring worldsheet symmetries

- 2d bosonic diffeomorphism

$$\sigma \to \tilde{\sigma}(\sigma, \tau)\,, \qquad \tau \to \tilde{\tau}(\sigma, \tau)$$

- fermionic κ symmetry

Fix these symmetries by suitably choosing a gauge.

Let us consider κ symmetry:

$$\Theta^I = (\theta, \eta)$$

$$\theta \leftrightarrow Q\,, \qquad \eta \leftrightarrow S$$

“S-gauge”

$$\eta = 0$$

Conformal gauge and S-gauge lead to a simple quadratic fermionic action (after “T-duality”). ...but bosonic propagator is *not simple*.

Disagreement with Bethe-Ansatz...!

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Other possibility:

$$\Gamma^+ \theta^I = 0$$

as for Green-Schwarz in flat space.

Left with 8 θ's and 8 η's.

Combine this with bosonic light-cone gauge

$$x^0 + x^1 = x^+ = \tau \,, \qquad \sqrt{-g} g^{\alpha\beta} = \mathrm{diag}(-z^2, 1/z^2)$$

Metsaev Thorn Tseytlin

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Very schematically

$$L_{\rm GS} \sim \dot{x}\dot{x}^* + \dot{z}^M \dot{z}^M + \frac{1}{z^4}\left(x' x'^* + z'^M z'^M\right)$$
$$+\eta\partial\eta + \theta\partial\theta + (\eta^2)^2 + \eta\partial\theta$$

Fermionic action is quadratic in θ and quartic η.

Bosonic propagator is simple (almost diagonal).

This is an encouraging property for higher loop computations.

- We have a 2d QFT theory problem
- Compute its partition function

$$Z_{\rm string} = \int \mathcal{D}[x, z, \theta, \eta] \; \exp\left(-\int L_{\rm GS}\right)$$

Expand bosonic fields around the semiclassical string solution

$$x = x_0 + \delta x \,, \qquad z^M = z_0^M + \delta Z^M$$

Read vertices and compute bosonic and fermionic Feynman diagrams at a given loop order

At one loop we simply need the fluctuation spectrum

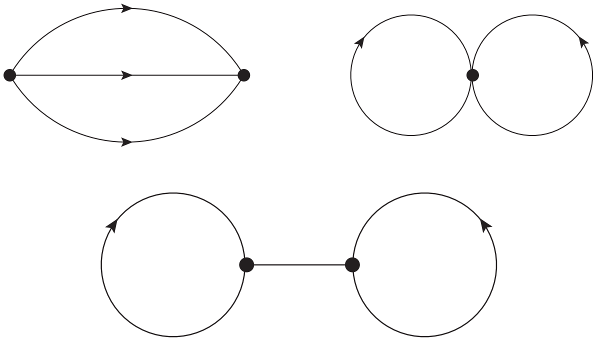
$$\mathcal{F}_{\text{one loop}} = \int d^2p \log\det(K_B\, K_F) \qquad \mathcal{Z}_{\text{one loop}} = e^{-\frac{V}{2\pi}\mathcal{F}_{\text{one loop}}}$$

Bosons and fermions conspire to give a finite answer

At two-loops for the computation of

$$\mathcal{F}_{\text{2loop}} \sim \log Z_{\text{string}}$$

we need to consider *all connected* Feynman diagrams



Possible integrals

$$\mathrm{I}\begin{pmatrix} a \\ m^2 \end{pmatrix} = \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p^2+m^2)^a}$$

This integral is UV divergent for $a = 1$ (and IR divergent for $m = 0$).

$$\mathrm{I}\begin{pmatrix} a_1 & a_2 & a_3 \\ m_1^2 & m_2^2 & m_3^2 \end{pmatrix} = \int \frac{d^2p\, d^2q\, d^2r}{(2\pi)^4} \frac{\delta^{(2)}(p+q+r)}{(p^2+m_1^2)^{a_1}\,(q^2+m_2^2)^{a_2}\,(r^2+m_3^2)^{a_3}}$$

Catalan constant:

$$\mathrm{I}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \mathrm{K}$$

Summing up:

$$\mathcal{F}_{2\text{loop}}(\ell) = -\mathrm{K} + \ell^2(\cdots) + \ell^4(\cdots) + \mathcal{O}(\ell^6)$$

All divergences cancel out!

The quantum superstring in AdS light cone gauge is finite

This proves the consistency of the light-cone superstring at the quantum level.

From the partition function we can extract the generalized scaling:

$$
\begin{aligned}
\mathrm{f}_2 &= -\mathrm{K} \\
&+ \ell^2\left(8\log^2\ell - 6\log\ell - \frac{3}{2}\log 2 + \ell^2\right) \\
&+ \ell^4\left(-6\log^2\ell - \frac{7}{6}\log\ell + 3\log 2\log\ell - \frac{9}{8}\log^2 2\right. \\
&+ \left.\frac{11}{8}\log 2 + \frac{3}{32}\mathrm{K} - \frac{233}{576}\right) + \mathcal{O}(\ell^6)
\end{aligned}
$$

In stupendous agreement with the Bethe-Ansatz prediction!

Leading logarithms and all loops results

Summarising we have

$$\mathrm{f}_2 = h_2(\ell) \log^2 \ell + h_1(\ell) \log \ell + h_0(\ell)$$

The coefficients $h_1(\ell)$ and $h_2(\ell)$ can be computed *exactly*.
At N-loop the expansion looks like

$$\mathrm{f}_N = f^{(N)}(\ell) \ \log^N \ell + f^{(N-1)}(\ell) \ \log^{N-1} \ell + \cdots$$

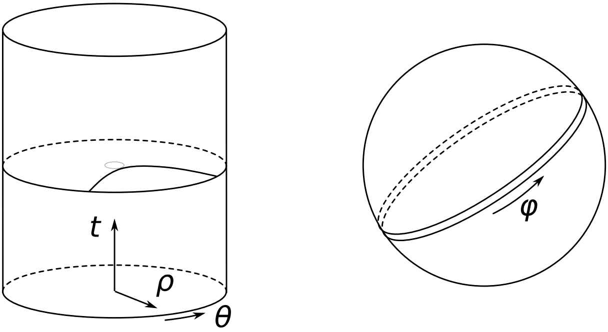
It is even possible to reconstruct the leading logarithm coefficient at *any* loop order!

Again in perfect agreement with an all-loop prediction from Bethe-Ansatz.

Computations were done in a more general background

$$\varphi = \nu\tau + w\sigma\ , \qquad w = \text{“winding”}$$

The solution with winding is a bended arc passing through AdS center



The partition function now depends on both momentum and winding:

$$\mathcal{F}(\nu, w)$$

It is again finite at two-loops

The exchange of ν and w is a symmetry: T-duality!

- We can extract a generalized scaling function in presence of winding
- Prediction confirmed at leading order from Bethe-Ansatz.

  Kruczenski-Tirziu

  One-loop/two-loops?

Finite size corrections

Compute finite spin subleading terms to cusp anomaly ($J = 0$).

The worldsheet's length is no longer infinite

$$L = \log S < \infty$$

Exact 1 loop result *Beccaria, Dunne, Forini, Pawellek, Tseytlin*

$$\gamma(g, S) = f(g) \log S + \gamma^{(0)}(g) + \frac{\gamma^{(-1)}(g)}{\log S} + \ldots$$

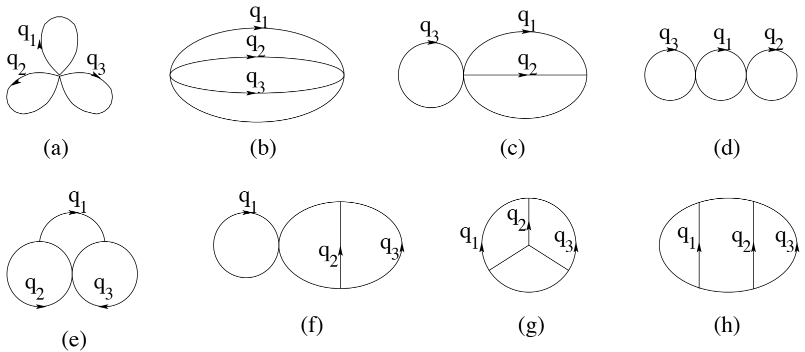
$$? \qquad \gamma^{(-1)} = 0 \qquad\qquad 2\,\text{loops}$$

in preparation

In agreement with Bethe-Ansatz expectations

Fioravanti, Grinza, Rossi

Going up...



q1
q2
q3
(a)
q1
q2
q3
(b)
q3
q1
q2
(c)
q3
q1
q2
(d)
q1
q2
q3
(e)
q1
q2
q3
(f)
q2
q1
q3
(g)
q1
q2
q3
(h)

The most feasible computation is the cusp anomaly ($J = 0$)

$$f(\lambda) = \sqrt{\lambda}\left(1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{\lambda} + \frac{a_3}{\lambda^{3/2}} + \frac{a_4}{\lambda^2} + \cdots\right)$$

Remarkable transcendentality properties

$$a_1 = -3\log 2\,, \quad a_2 = -\mathrm{K}$$
$$a_3 = -\frac{1}{32}\left(27\zeta(3) + 96\,\mathrm{K}\log 2\right)$$
$$a_4 = -\frac{1}{16}\left(84\beta(4) + 81\zeta(3)\log 2 + 32\,\mathrm{K}^2 + 144\,\mathrm{K}\log^2 2\right)$$

Degree of transcedentality

$$[\log 2] = 1 \quad [K] = 2 \quad [\zeta(n)] = n \quad [\beta(n)] = n; \qquad \beta(2) = \mathrm{K}$$

Conclusion

- The light-cone superstring computation solves a long standing discrepancy between string and Bethe-Ansatz.
- It provides a highly non trivial check of quantum integrability beyond one-loop string semiclassical level.
- Viceversa it provides an important consistency check of all-loop Bethe-Ansatz.