The ABC of AGT

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Outline

Prologue

[Seiberg–Witten'94] taught us how the low energy dynamics of $\mathcal{N}=2$ theories is encoded in the SW curve.

[Witten'97] proposed an M theory construction for linear quivers: the $4d$ gauge theory comes from a compactification of the $6d$, $(2,0)$ theory of A_{N-1} type.
The SW curve is realized as an N sheeted branched covering of a punctured complex sphere or torus, with a canonical choice of SW differential $\Rightarrow$ Full characterisation of the gauge theory in particular under S–duality transformations!

Example: compactify the $SU(N), 6d, (2,0)$ theory on a torus and understand S–duality in $SU(N)$ $\mathcal{N}=4$ gauge theory as $\tau \rightarrow \frac{1}{\tau}$.

$\cdots$ building on this ideas [Gaiotto'09]: compactify on more general surfaces to construct $\mathcal{N}=2$ super-conformal gauge theories $\rightarrow$ Generalized S-duality transformations!

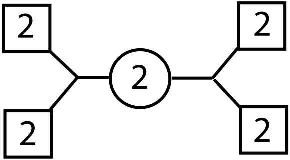
Are Lagrangians out of fashion?

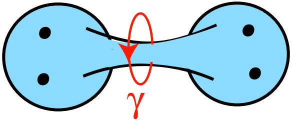
$\mathcal{N} = 2$ theories

[Gaiotto'09] constructs superconformal $\mathcal{N} = 2$ theories denoted by $\mathcal{T}_{g,n}$, by compactifying the 6d $(2,0)$ theory of A_{N-1} type on $C_{g,n} \times \mathbb{R}^4$ (world volume of the M_5 branes).
Space of inequivalent superconformal gauge theories is:

$$\mathcal{M}_{g,n} = \mathbb{T}^{g,n}/\Gamma_{g,n},$$

where $\mathbb{T}^{g,n}$ is the space of coupling constants and the $\Gamma_{g,n}$ action $\to$ S-duality group.
The prototypical example is: $SU(2), N_f=4$

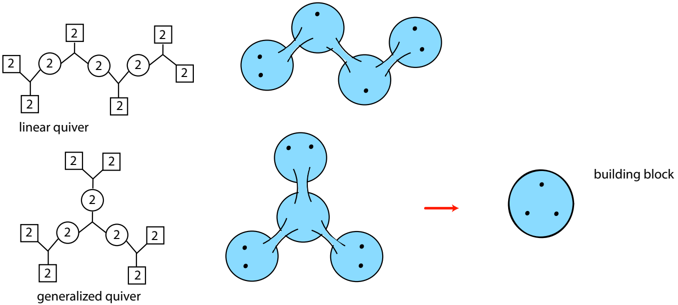




The Seiberg-Witten curve $\Sigma_{SW} : x^2 = \phi_2(z)$ is the double cover of $C_{g,n}$. $\phi_2(z)$ has double poles at the punctures with coeff m_i and $\lambda_{SW} = x(z)dz$.

More complicated quivers:

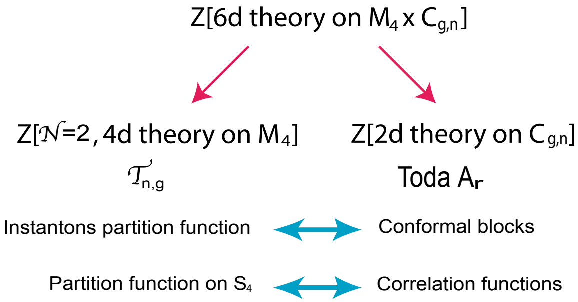
$$SU(2)^3, N_f = 6$$



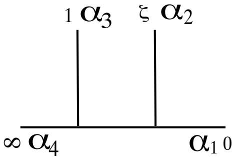
- Different "pairs of pants" decompositions correspond to different weekly coupled Lagrangian descriptions of the same theory.
- The sphere with 3-punctures, denoted by $\mathcal{T}_{0,3} = T_2$ is the building block.
- For higher rank groups punctures carry generic representations of $SU(N)$.
- The sphere with 3 full punctures, the T_N theory has no Lagrangian description.

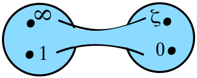
The AGT conjecture

[AGT'09,Wyllard'09] is a statement about the equivalence of two distinct KK reductions:



More precisely the $SU(2)$, $N_f = 4$ theory is related to the Liouville 4-point correlator of chiral primaries $V_\alpha(z_i) = e^{2\alpha\phi(z_i)}$ with $\Delta = \alpha(\mathcal{Q} - \alpha)$, $\mathcal{Q} = b + \frac{1}{b}$:





$$\langle \alpha_4 | V_{\alpha_3}(1) V_{\alpha_2}(\zeta) | \alpha_1 \rangle = \int d\sigma\, C(\alpha_4^*, \alpha_3, \sigma) C(\sigma^*, \alpha_2, \alpha_1) |\zeta^{\Delta_\sigma - \Delta_{\alpha_2} - \Delta_{\alpha_1}} \mathcal{F}_{\alpha_4\alpha_3\sigma\alpha_2\alpha_1}(\zeta)|^2$$

$2d$CFT	$4d$ gauge theory
chiral conformal block : $\mathcal{F}_{\alpha_4\alpha_3\sigma\alpha_2\alpha_1}(\zeta)$	$Z_{\text{inst}}(\sigma, \zeta) \sim e^{-\frac{1}{\hbar^2} F_0 + \cdots}$
3point function : $C(\alpha_1, \alpha_2, \alpha_3)$	$Z_{1-\text{loop}}$
cross ratio ζ	$e^{2\pi i \tau_{UV}}$
level k	instanton number
external momenta α_i	masses m_i
internal momentum σ	coulomb branch a
b	ϵ_1
$1/b$	ϵ_2

Surface operators

An M_2 brane can attach to M_5 by an open boundary surface $\mathcal{S}$ sourcing the 2-form B. Different ways of embedding $\mathcal{S}$ in $C_{g,n} \times \mathbb{R}^4$ give rise to different operators in the 4d gauge theory.

- Surface operator: $\mathcal{S}$ is embedded in $\mathbb{R}^4$ and localized at a point z on $C_{g,n}$.
- Loop operator: M_2 wraps a one-cycle γ in $C_{g,n}$.
- Vertex operator: M_2 wraps a two-cycle junction between loops operators.

[AGGTV'09] conjecture that surface operators correspond in the Liouville theory to the insertion of primary fields $V_{-b/2}$ satisfying the null state condition $(L_{-1}^2 + b^2 L_{-2})V_{-b/2} = 0$.

Surface operators

In [AGGTV'09] they conjeture that, in the semiclassical limit:

$$Z_{\mathrm{null}} = \frac{\langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(\zeta) V_{-b/2}(z) | \alpha_4 + \frac{b}{2} \rangle}{\langle V_{\alpha_1} | V_{\alpha_2}(1) V_{\alpha_3}(\zeta) | \alpha_4 \rangle} = \mathrm{e}^{-\frac{b}{\hbar} G_0(z) + \cdots} .$$

A WKB-type argument and the fact that $V_{-b/2}$ is degenerate implies that:

$$(\partial_z G_0)^2 + \phi_2(z) = 0$$

and thus:

$$G_0(z) = \int^z \lambda_{SW}(z) = \int^z x(z')\, \mathrm{d}z'$$

so in the semiclassical limit the effect of the surface operator is encoded in the SW curve.

How to go beyond this?

On the CFT side we can:

- go beyond the semiclassical limit
- insert more degenerate fields

we will see that the SW curve encodes all this as well.

The T_2 theory

Let's consider a surface operator in the $T_2 = \mathcal{T}_{0,3}$ theory aka the 3-point function with a degenerate insertion in Liouville:

$$Z_{\text{null}}(z) = \frac{\langle \alpha_1 | V_{\alpha_2}(1) V_{-b/2}(z) | \alpha_3 + \frac{b}{2} \rangle}{\langle \alpha_1 | V_{\alpha_2}(1) | \alpha_3 \rangle} .$$

A famous result in Liouville [BPZ'83] is that:

$$Z_{\text{null}}(z) = z^{b\alpha_3}(1-z)^{b\alpha_2} \, {}_2F_1(A_1, A_2; B_1; z) ,$$

with:

$$A_1 = b(\alpha_1 + \alpha_2 - \alpha_3) , \quad A_2 = b(\alpha_1 + \alpha_2 + \alpha_3 - \mathcal{Q}) , \quad B_1 = 2 b \alpha_1 .$$

The semiclassical expansion is obtained by rescaling $\alpha_i \to \alpha_i/\hbar$ and Taylor expanding in $\hbar$ and $\mathcal{Q} = b + 1/b$:

$$Z_{\text{null}}(z) = \exp\left\{ \frac{b}{\hbar} G_0(z) + b^2 G_1(z) + b^3 \hbar \, G_2(z) + \mathcal{O}(\hbar^2) \right\} ,$$

with

$$G_i(z) = \sum_{n \geq 0} G_i^n(z) \mathcal{Q}^n .$$

The T_2 theory

For $k > 1$ insertions of degenerate fields $V_{-b/2}$ we find:

$$Z_{\text{null}}(z_1, \cdots, z_k) = \frac{\langle \alpha_1 | V_{\alpha_2}(1) V_{-b/2}(z_1) \cdots V_{-b/2}(z_k) | \alpha_3 + k\frac{b}{2} \rangle}{\langle \alpha_1 | V_{\alpha_2}(1) | \alpha_3 \rangle} =$$

$$= \left(\prod_{i=1}^{k} z_i^{b\alpha_1} (1 - z_i)^{b\alpha_2} \right) {}_2F_1^{\beta}(A_1, A_2; B_1; z_1, \ldots, z_k),$$

so in this case

$$Z_{\text{null}}(z_1, z_2) = \exp\left\{ \frac{b(G_0(z_1) + G_0(z_2))}{\hbar} + b^2 G_1(z_1, z_2) + \mathcal{O}(\hbar) \right\}$$

with

$$G_i(z_1, \cdots, z_k) = \sum_{n \geq 0} \mathcal{Q}^n G_i^n(z_1, \cdots, z_k).$$

The topological recursion

Since from [AGGTV'09]

$$G_0(z) = \int^z \lambda_{SW}(p)$$

we expect that the $G_i(z_1, \cdots z_k)$ are related to higher differentials $\mathcal{W}_k^{(g)}(p_1, \cdots, p_k)$ on the SW curve Σ_{SW}; Generate higher differentials with the Matrix Models Topological Recursion [EO'07].

Ingredients:

- The branch points $x(q_i)$ on $\Sigma: H(x,y) = 0$, satisfying $\frac{dH(q_i)}{dy} = 0$
- The meromorphic differential $\lambda(x)$ on Σ and the Bergmann kernel $B(p,q)$ on Σ.

The topological recursion generates higher differentials:

$$\mathcal{W}_{h+1}^{(g)}(p, p_1 \ldots, p_h) = \sum_{q_i} \operatorname*{Res}_{q=q_i} F\left[\mathcal{W}_{h'}^{g'}, \lambda\right],$$

with $F\left[\mathcal{W}_{h'}^{g'}, \lambda\right]$ a complicated function of $\mathcal{W}_{h'}^{(g')}$ with $g' \leq g$, $h' \leq h$ and of $\mathcal{W}_{h+1}^{g-1}$.

The topological recursion:

We conjecture that

$$Z_{\text{null}}(z)\big|_{\mathcal{Q}=0} = \exp\Big[\sum_{g,k}\hbar^{2g-2+k}\frac{1}{k!}A_k^{(g)}(z,\cdots,z)\Big]$$

$$= \exp\Big[\frac{1}{\hbar}A_1^{(0)}(z)+\frac{1}{2!}A_2^{(0)}(z,z)+\hbar\left(A_1^{(1)}(z)+\frac{1}{3!}A_3^{(0)}(z,z,z)\right)+\cdots\Big].$$

with

$$A_k^{(g)}(z_1,\cdots,z_k) = \int^{z_1}\cdots\int^{z_k} \mathrm{d}p_1\cdots\mathrm{d}p_k\,\mathcal{W}_k^{(g)}(p_1,\cdots,p_k)\,.$$

By comparing to the CFT expressions we checked that

$$G_0(z)\big|_{\mathcal{Q}=0} = A_1^{(0)}(z),\quad G_1(z)\big|_{\mathcal{Q}=0} = \frac{1}{2!}A_2^{(0)}(z,z),\quad G_2(z)\big|_{\mathcal{Q}=0} = A_1^{(1)}(z)+\frac{1}{3!}A_3^{(0)}(z,z,z)\,.$$

For multiple insertions for instance, we have

$$G_1(z_1,z_2)\big|_{\mathcal{Q}=0} = A_2^{(0)}(z_1,z_2)+\tfrac{1}{2}(A_2^{(0)}(z_1,z_1)+A_2^{(0)}(z_2,z_2))\,.$$

The topological recursion approach:

yet another way of solving Liouville theory$\cdots$new viewpoints can be useful (Toda?)

- $Z_{\rm null}(z)$ has the expansion of a determinant. [DV'09] provided a matrix model realization of T_2 and we consistently find that a degenerate field corresponds to a $Tr\frac{1}{x-M}$ insertion.
- $Z_{\rm null}(z)$ is related to a Baker-Akhiezer function and satisfies a Hirota equation (for the multi-component KP hierarchy) [EO'07]
- we studied also the $\mathcal{T}_{0,4}$ theory; more interesting. Quasi-modularity of the $\mathcal{W}_h^{(g)}$
- use the β ensemble for generic $\mathcal{Q}$

For sure useful to better understand surface operators in the gauge theory $\rightarrow$

Topological strings

Topological strings come in two types *A&B* models, from the two ways of twisting the $(2, 2)$ theory to get a BRST–like operator.

A model describes Kähler structure deformations while the B model complex structure deformations.

Topological strings are highly solvable theories, major role of dualities, *A&B* models are related by mirror symmetry.

Topological strings come naturally into the game because, for toric Calabi–Yau targets:

- A they can be used to engineer $\mathcal{N} = 2$ gauge theories and to obtain the instantons partition function $\rightarrow 4d$
- B all the non trivial content of the theory is encoded in the mirror curve (SW curve) $\rightarrow 2d$

Topological A–Model

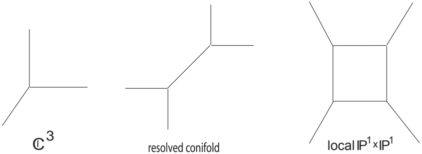
Topological type A string theory is a theory of maps from a Riemann surface to a CY target X. The free energy, enumerating maps $f : \Sigma_g \mapsto X$, is defined as:

$$\log Z^{\text{top}}(X) = F(X) = \sum_{g,d} g_s^{2g-2} N_{g,d}\, Q^d$$

- $Q = \mathrm{e}^{-T}$ is the closed modulus, T is the Kähler parameter measuring sizes in the target in units of α'.
- $N_{g,d}$ are Gromov–Witten invariants, counting maps in the class g, d.

Toric Calabi–Yau's

- Toric CY 3folds are $\mathbb{R}_+ \times T^2$ fibration over $\mathbb{R}^3$ and can be can be obtained by gluing $\mathbb{C}^3$ patches.
- For each $\mathbb{C}^3$ the degeneration locus of the T^2 fiber is encoded in a trivalent planar diagram. Gluing vertices we obtain the toric graph:



- To each $\mathbb{C}^3$ we associated the refined vertex [IKV'07]:

$$C_{\lambda\mu\nu}(q,t)=\left(\frac{q}{t}\right)^{\frac{\|\mu\|^2+\|\nu\|^2}{2}} t^{\frac{\kappa(\mu)}{2}} P_{\nu^t}(t^{-\rho};q,t)\sum_{\eta}\left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu})s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})$$

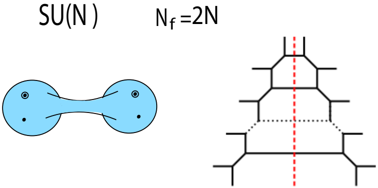
where $q=e^{\epsilon_1}$, $t=e^{-\epsilon_2}$ and $P_\nu(\mathbf{x};q,t)$ is the Macdonald function.

- By gluing with appropriate rules one obtains the refined topological string partition function. For instance for the resolved conifold we get:

$$Z^{\rm ref}_{\rm top}(X_{\rm res})=\sum_\mu C_{00\mu^t}(q,t)C_{00\mu}(t,q)(-1)^{|\mu|}Q^{|\mu|}$$

Toric engineering

$\mathcal{N}=2$ gauge theories can be engineered by IIA compactification on toric CY's. For instance:



In particular

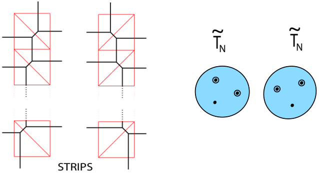
$$Z^{\text{top}}_{\text{ref}} = Z^{\text{inst}}_5$$

where Z^{inst}_5 is the LMNS partition function formulated on $\mathbb{R}^4 \times S^1$. In the limit $R \to 0$ one recovers the 4d partition function:

$$\lim_{R\to 0} Z^{\text{top}}_{\text{ref}} = Z^{\text{inst}}$$

Toric approach to Toda 3points functions?

Now consider the following degeneration:



The $\widetilde{T}_N$ theory is related via AGT to the Toda 3-point function of $V_\alpha = e^{<\alpha,\phi>}$ with $\phi = \phi_i e_i$, $\mathcal{Q} = (b + 1/b)\rho$ for $\alpha_3 = \kappa = \chi \Lambda_{1,N}$, that reads:

$$\langle V_{\alpha_1}(z_1\bar{z}_1) V_{\alpha_2}(z_2\bar{z}_2) V_\kappa(z_3\bar{z}_3)\rangle = \frac{C_{\alpha_1\alpha_2\kappa}}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}$$

Recently [FL'05-'07] computed:

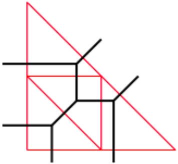
$$C_{\alpha_1\alpha_2\kappa} = \prod_{ij} |\Gamma_b(\frac{\chi}{N+1} + < \alpha_1 - \mathcal{Q}, h_i > - < \alpha_2 - \mathcal{Q}, h_j >)|^2$$

where $\Gamma_b(z) = \Gamma_2(z|b, 1/b)$ is the log of the double Barnes function and we used the $V_\alpha \to f(\alpha) V_\alpha$ freedom. We checked that:

$$\lim_{R\to 0} Z^{\rm ref}_{\rm top}(\widetilde{T}_N) = \prod_{ij} \Gamma_b(\frac{\chi}{N+1} + < \alpha_1 - \mathcal{Q}, h_i > - < \alpha_2 - \mathcal{Q}, h_j >).$$

Toric approach to 3points functions?

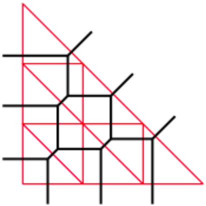
A long standing problem in Toda theories is the computation of $C_{\alpha_1 \alpha_2 \alpha_3}$, which via AGT, is related to the T_N theory.
[BBT'09] proposed that the toric geometry engineering the 5d lift of T_N is the blow up of the $\mathbb{C}^3/[\mathbb{Z}_N \times \mathbb{Z}_N]$ orbifold.
For T_2 the relevant geometry is:



we checked that:

$$\lim_{R\to 0} Z^{\text{ref}}_{\text{top}}(\widetilde{T}_2)\text{''} = \text{''} \lim_{R\to 0} Z^{\text{ref}}_{\text{top}}(T_2) = \Gamma_b(\alpha_1+\alpha_2+\alpha_3-\mathcal{Q})\Gamma_b(\alpha_2+\alpha_3-\alpha_1)$$
$$\times\Gamma_b(\alpha_3+\alpha_1-\alpha_2)\Gamma_b(\alpha_1+\alpha_2-\alpha_3) =$$
$$= C_{\alpha_1\alpha_2\alpha_3} = \text{Liouville 3point function}$$

...so one may hope that the T_3 geometry:



is related to the Toda A_2 3-point function and so on.
But there are many issues:

- how to define the 4d limit?
- There is a non-trivial Kähler moduli space, need to choose a duality frame, the vertex picks the large radius patch. Is it the right one?

At least the number of parameters is matching:

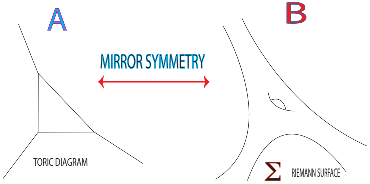
$$\#\text{Toda 3point function} = \frac{(N+4)(N-1)}{2} = \# T_N$$

B model interpretation

The mirror of a toric CY is given by a cone over a Riemann surface in $\mathbb{C}^* \times \mathbb{C}^*$:

$$Y_t : w^+ w^- = H(x, y; t)$$

with $w^+, w^- \in \mathbb{C}$ and $x = \exp(u), y = \exp(v) \in \mathbb{C}^*$ [Hori–Vafa].
The fiber degenerates to two lines over the mirror curve $H(x, y; t) = 0$:



Open B-model amplitudes ends on B-branes that wrap holomorphic curves in Y_t with trivial bundles, defined by:

$$w' = 0 = H(x_0, y_0; t).$$

The open moduli space corresponds to deformations of the B-brane in Y_t which are parameterized by the points $(x_0, y_0) \in H(x, y, t)$. The open moduli space coincides with the mirror curve.

- The B model disk amplitude [AV'01,AKV'01] reads:

$$A_1^{(0)}(z) = \int^z \lambda(p) = \int^z \log(y(p;t))\frac{\mathrm{d}p}{p}.$$

This is precisely the surface operator in the semiclassical limit [AGGTV'09]!

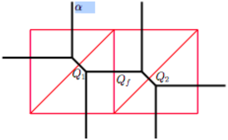
- Higher open B model amplitudes are given by the Remodeling approach [BKMP'07]:

$$A_h^{(g)}(z_1,\cdots z_h) = \int^{z_1}\cdots\int^{z_h} \mathcal{W}_h^{(g)}(p_1\cdots p_h)$$

By simply taking the $R \to 0$ limit we obtain the $G_i(z_1\cdots z_k)$ from the $A_h^{(g)}(z_1,\cdots z_h)$.
This is an explicit MIRROR B model proof that:
SURFACE OPERATORS ARE REALISED AS TORIC BRANES

A model approach

Open topological A model amplitudes ends on L which is a Lagrangian submanifold($\omega|_L = 0$). In the Toric case L has $\mathbb{C} \times S^1$ topology[AV'01].
For the $\widetilde{T}_2$ geometry:



the strip ($\mathcal{Q} = 0$) computation leads to the following open amplitude:

$$Z_\alpha = s_\alpha(q^\rho) \prod_k \frac{\left(1 - q^k Q_1\right)^{C_k(\alpha,\bullet)} \left(1 - q^k Q_1 Q_2 Q_f\right)^{C_k(\alpha,\bullet)}}{\left(1 - q^k Q_1 Q_f\right)^{C_k(\alpha,\bullet)}}.$$

for a single toric brane α is a column of length n and:

$$C_k(\alpha, \bullet) = \begin{cases} 1 & \text{for } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

and we obtain:

$$Z_{(n)}(Q_1, Q_2, Q_f, q) = \prod_{k=1}^{n} \frac{\left(1 - q^k Q_1\right)\left(1 - q^k Q_1 Q_f Q_2\right)}{\left(1 - q^k\right)\left(1 - q^k Q_1 Q_f\right)}.$$

A model approach

We can now package the contributions from columns of length n into the generating function:

$$Z_{\text{open}}(z) = \sum_{n=0}^{\infty} z^n Z_{(n)}(Q_1, Q_2, Q_f, q).$$

With the AGT ($Q_i \to \alpha_i$, etc) dictionary we checked that:

$$\lim_{R\to 0} Z_{\text{open}}(z) = Z_{\text{null}}(z)\Big|_{\mathcal{Q}=0}$$

- the insertion of 2 surface operators correspond to reps with 2 columns and so on. Describes also other types of degenerate fields.
- can be refined (qt-hypergeomertic [Kaneko'96])
- can be extended to $\widetilde{T}_N$ geometries and to $SU(N)$ theories.

Summary & Outlook

- we checked explicitly that surface operators are realized by toric branes (proposed also by Gukov).
- the A model is non-perturbative in $\hbar, \mathcal{Q}$ but perturbative in the instanton number just like the CFT pertrubative result.
- the B model is perturbative in $\hbar, \mathcal{Q}$ but exact in the complex structure parameters, this fact allows the study of the modularity property of the amplitudes. Mirror Symmetry allows to move in the moduli space of complex structure.
- toric approach maybe useful also for the computation of Toda 3-point functions and to better understand punctured surfaces.
- computation&classifications of surface operators
- surface operators related to $V_{mb+n/b}$ and more general..
- relation to WZW