Non-Abelian vortices and lumps; their moduli spaces and substructures

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Theories of fundamental interactions
Perugia
June 25, 2010

In collaboration with: Roberto Auzzi, Minoru Eto, Toshiaki Fujimori, Yunguo Jiang, Kenichi Konishi, Takayuki Nagashima, Muneto Nitta, Keisuke Ohashi and Walter Vinci.

arXiv:1005.0557 SBG
arXiv:1002.0850 Konishi,SBG
arXiv:0906.0021 SBG
arXiv:0905.3540 Eto,Fujimori,SBG,Konishi,Nagashima,Nitta,Ohashi,Vinci
arXiv:0903.4471 Eto,Fujimori,SBG,Konishi,Nagashima,Nitta,Ohashi,Vinci
arXiv:0809.2014 Eto, Fujimori, SBG, Nitta, Ohashi

Plan of the talk

Introductory part

→ ○ motivation
- crash course on vortices
- summary of recent results in the field

Results

- SO, USp theories
- NLσM lumps
- fractional vortices
- Chern-Simons vortices

Motivation

- Confinement of quarks in Yang-Mills theory ('t Hooft-Mandelstam)
- Electric-Magnetic duality (Dirac, Montonen-Olive)
- Non-Abelian monopoles and GNOW-duality (Goddard-Nuyts-Olive-Weinberg)
- Condensed matter systems – quantized vortices
- Fractional quantum Hall effect – Chern-Simons theory
- Neutron stars
- Cosmic strings
- etc.

“Dynamical Abelianization”?

- Softly broken $\mathcal{N}=2$ pure $SU(2)$ in Seiberg-Witten theory $\longrightarrow$ Abelian
- Generic $\mathcal{N}=2$ super-Yang-Mills (SYM) with quarks $\longrightarrow$ non-Abelian

Seiberg 1994, Seiberg-Witten 1994, Argyres-Plesser-Seiberg 1996

Hanany-Oz 1996, Carlino-Murayama-Konishi 2000

Question: magnetic monopoles of QCD are of Abelian or non-Abelian type?

Non-Abelian monopoles

Symmetry-breaking pattern:

$$G \to H \neq U(1) \ , \qquad (1)$$

Field strength tensor: $F_{ij} \sim \epsilon_{ijk} \frac{x_k}{r^3} (\beta \cdot T_{\text{Cartan}})$.
Charge quantization:

$$2\beta \cdot \alpha = \mathbb{Z} \ , \qquad (2)$$

α is the root vector of H. Solution: β is any weight vector of the dual group $\tilde{H}$ which has root vectors

$$\alpha^\star = \frac{\alpha}{\alpha \cdot \alpha} \ . \qquad (3)$$

The GNOW conjecture:
non-Abelian monopoles form multiplets of the dual group $\tilde{H}$.

H	$U(N)$	$SU(N)$	$SO(2N)$	$SO(2N+1)$
$\tilde{H}$	$U(N)$	$\frac{SU(N)}{\mathbb{Z}_N}$	$Spin(2N)$	$USp(2N)$

Unfortunately, there are some obstacles:

1. topological obstruction
2. non-normalizable zero-modes – going to zero as $r^{-\frac{1}{2}}$

$\Rightarrow$ we cannot quantize the non-Abelian monopole

The transformations in H and $\tilde{H}$, respectively are non-local.
Problematic: Coulomb phase and (dual) Higgs phase.
Strategy:

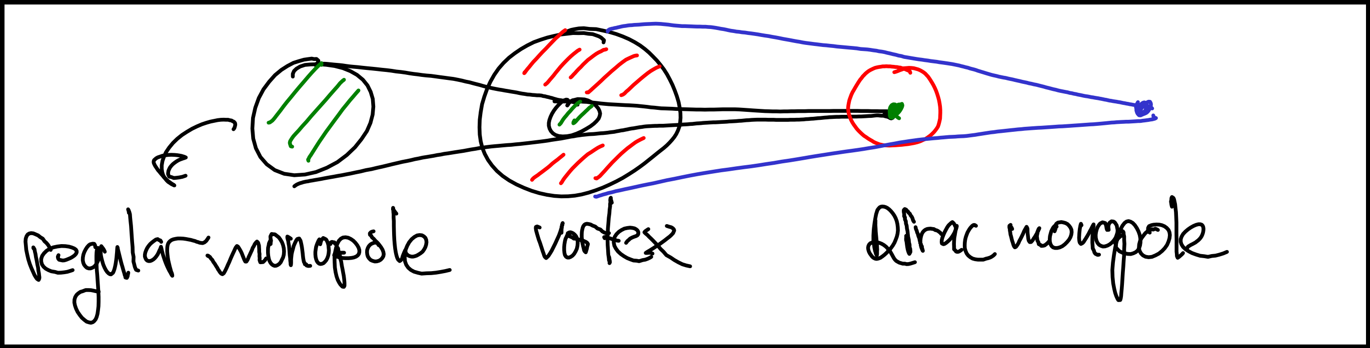
$$H_{\text{Higgs phase}} \Leftrightarrow \tilde{H}_{\text{confinement phase}}$$

Benchmark model

$$G \underset{\text{monopole}}{\overset{\Lambda}{\to}} H \underset{\text{vortex}}{\overset{\mu}{\to}} \mathbb{1} \ , \quad \Lambda \gg \mu \ . \tag{4}$$

Exact homotopy sequence:

$$\ldots \to \pi_2 (G) \to \pi_2 \left(\frac{G}{H} \right) \to \pi_1 (H) \to \pi_1 (G) \to \ldots \tag{5}$$



Example:

$$\mathbb{1} = \pi_2 (SU(3)) \to \pi_2 \left(\frac{SU(3)}{SU(2) \times U(1)} \right) \tag{6}$$

$$\to \pi_1 (SU(2) \times U(1)) \to \pi_1 (SU(3)) = \mathbb{1} \ .$$

Paradox?

Complete system

$$G \to \mathbb{1} \;, \tag{7}$$

$\Rightarrow$ no monopoles

$$\pi_1(G) = \mathbb{1}, \tag{8}$$

$\Rightarrow$ no vortices

Solution:

Vortices confine the monopoles

Idea: vortex transformation = monopole transformation

Auzzi-Bolognesi-Evslin-Konishi 2004

Auzzi-Bolognesi-Evslin-Konishi-Yung 2003

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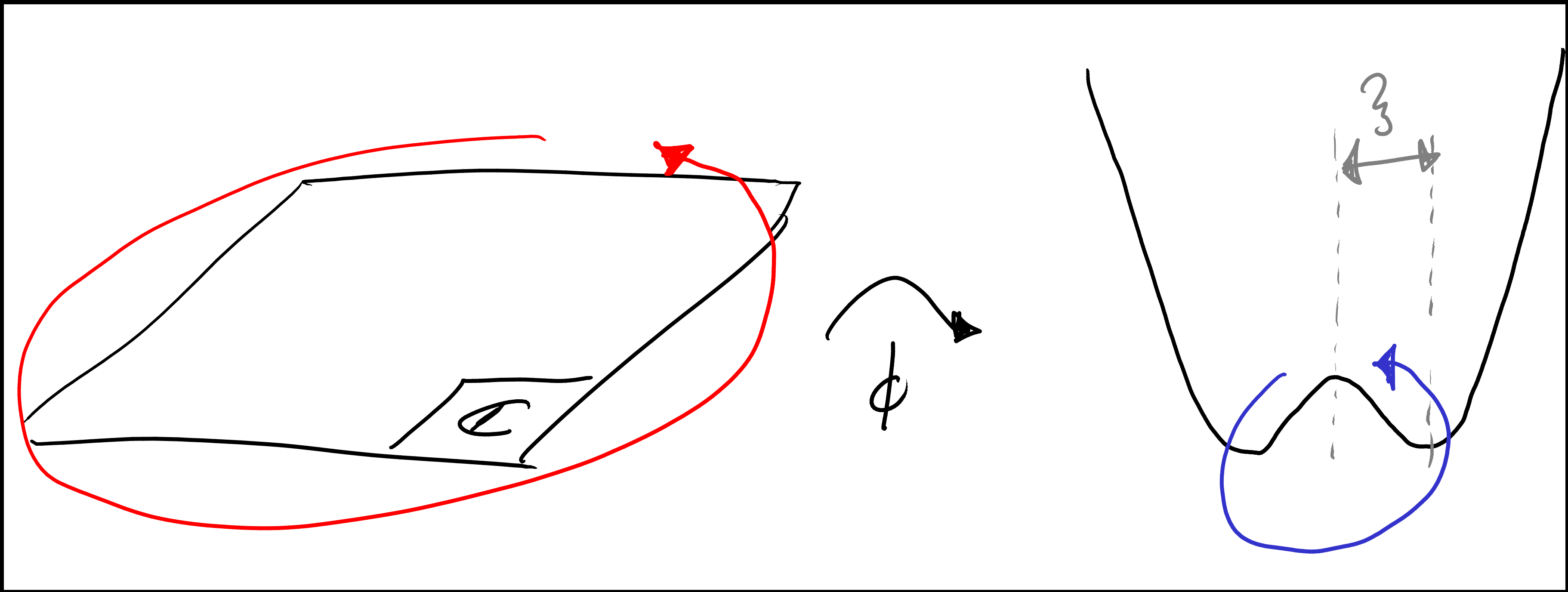
Abrikosov-Nielsen-Olesen vortex



$\Rightarrow \pi_1(S^1) = \mathbb{Z} \ni k$: winding number = vorticity

ξ : Fayet-Iliopoulos parameter – theory on the Higgs branch

Derrick's theorem

No finite energy scalar field configuration in more than one spatial dimension, other than the vacuum can have a stationary point

- $d = 1$: domain wall – OK
- $d = 2$: vortex – stabilized by
  - flux
  - $V = 0 \Rightarrow$ sigma model lumps – harmonic maps
  - $J \neq 0$ – Q-lumps

The vortex stabilized by magnetic flux

Competition of forces:

scalar field attractive force	$\longleftrightarrow$	magnetic field repulsive force

$\beta \equiv \frac{m_{\text{Higgs}}}{m_\gamma}$ classifies the vortices into:

- $\beta < 1$: type I – not experimentally observable as the flux attracts and breaks the superconducting phase
- $\beta > 1$: type II – Abrikosov lattice
- $\beta = 1$: BPS – supersymmetry preserving

Abrikosov lattice

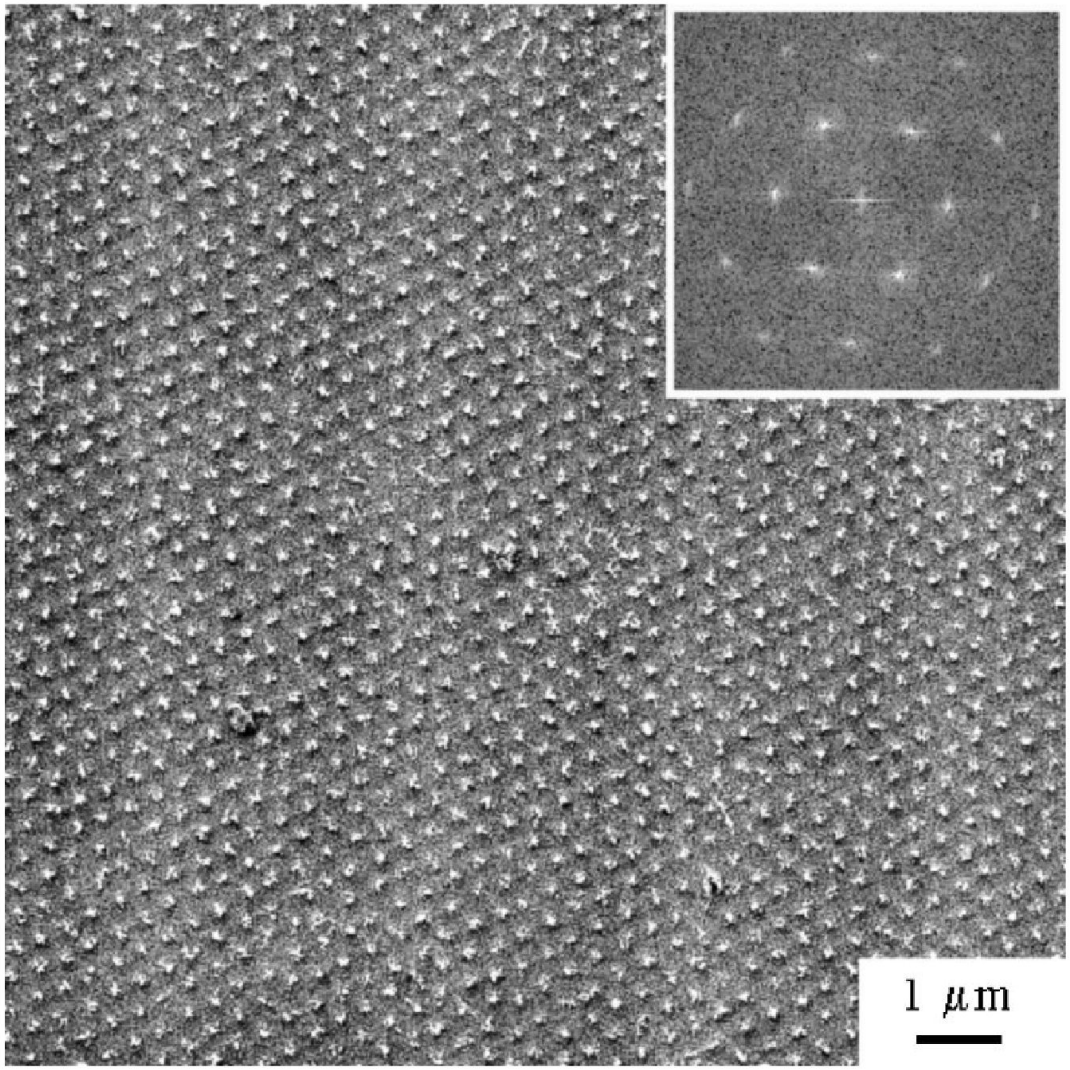


FIG. 2. Triangular vortex lattice at magnetic field 200 Oe in MgB_2 single crystal. Inset: FFT pattern in an arbitrary scale.

Non-Abelian embedding

$U(N)$ theory with $N_{\rm F} = N$ flavors:

$$q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1N} \\ q_{21} & q_{22} & & q_{2N} \\ \vdots & & \ddots & \vdots \\ q_{N1} & \cdots & \cdots & q_{NN} \end{pmatrix} , \quad \langle q \rangle = \mathbf{1}_N \tag{9}$$

Transformation

$$q \to U_{\rm color} q U^\dagger_{\rm flavor} , \tag{10}$$

$U_{\rm flavor} = U_{\rm color}$ global symmetry.
Embedding of ANO in $U(N)$ theory:

$$q = \begin{pmatrix} q^{\rm ANO} & 0 & \cdots & 0 \\ 0 & \sqrt{\xi} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \sqrt{\xi} \end{pmatrix} , \quad A_i = \begin{pmatrix} A_i^{\rm ANO} & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} .$$

A whole family of solutions appears!

Non-Abelian moduli

Color-flavor rotation:

$$q = U \begin{pmatrix} q^{\mathrm{ANO}} & 0 & \cdots & 0 \\ 0 & \sqrt{\xi} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \sqrt{\xi} \end{pmatrix} U^{\dagger} \; , \qquad (11)$$

$$A_i = U \begin{pmatrix} A_i^{\mathrm{ANO}} & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} U^{\dagger} \; .$$

$U \in G_{\mathrm{color+flavor}}$ global symmetry.

Non-Abelian moduli parametrize:

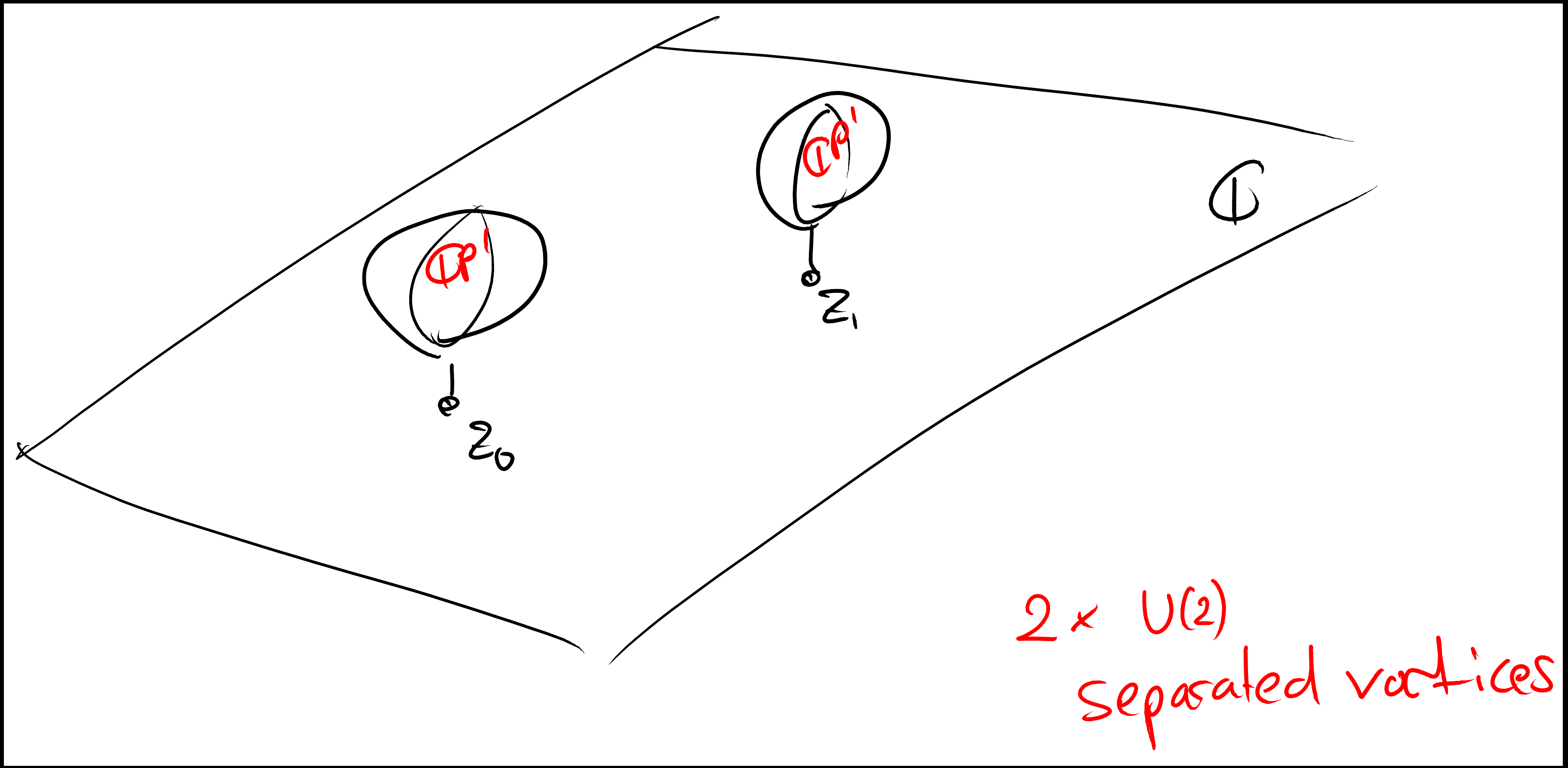
$$\frac{SU(N)}{SU(N-1) \times U(1)} \simeq \mathbb{C}P^{N-1} \; . \qquad (12)$$

$\Rightarrow$ effective world-sheet symmetry.

Moduli space the non-Abelian vortex

$U(N)$ theory:

$$\mathcal{M}_k = \frac{\left(\mathbb{C}\times\mathbb{C}P^{N-1}\right)^k}{\mathfrak{S}_k}$$



Hanany-Tong 2003, Auzzi-Bolognesi-Evslin-Konishi-Yung 2003, Shifman-Yung 2004

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Summery of recent results

- A duality between 4 dim SYM and a 2 dim non-linear σ model, realized by the non-Abelian vortex

Hanany-Tong 2003, Shifman-Yung 2004

- $SO(5) \to U(2) \to \mathbb{1}$: regular monopoles are confined by $k = 2$ vortices.

$k = 2$ moduli space of $U(2)$ vortices:

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \to W\mathbb{C}P^1_{2,1,1} \ ,$$

Hashimoto-Tong 2005, Auzzi-Shifman-Yung 2005, Eto-Konishi-Marmorini-Nitta-Ohashi-Vinci-Yokoi 2006

The vortex-monopole system transforms under the $\mathbf{3} + \mathbf{1}$ representation of $SU(2)$.

Weinberg 1982, Lee-Weinberg-Yi 1996, Kampmeijer-Slingerland-Schoers-Bais 2008

Group theory of vortices: single vortex $\Box$: k vortices $\prod_{i=1}^{k} \Box$.

In preparation

- The full moduli space:

Eto-Isozumi-Nitta-Ohashi-Sakai 2005

- Reconnection of cosmic strings

Hanany-Hashimoto 2004, Hashimoto-Tong 2005,
Eto-Hashimoto-Marmorini-Nitta-Ohashi-Vinci 2006

- D-brane solitons in field theory, e.g. an instanton-monopole-vortex-domain wall system

Tong 2005, Eto-Isozumi-Nitta-Ohashi-Sakai 2006, Shifman-Yung 2007

- A Seiberg-like duality of non-Abelian semi-local vortices

Eto-Evslin-Konishi-Marmorini-Nitta-Ohashi-Vinci-Yokoi 2007

- Non-BPS non-Abelian vortex interactions: distance-dependent forces (type I/I* and type II/II*)

Auzzi-Eto-Vinci 2007

- Non-Abelian vortices on a torus:

Lozano-Marques-Schaposnik 2007

- Non-Abelian vortices in dense QCD:

Nakano-Nitta-Matsuura 2007, Eto-Nitta 2009, Eto-Nakano-Nitta 2009, Eto-Nitta-Yamamoto 2009, Yasui-Itakura-Nitta 2001

- A model of non-Abelian vortices without dynamical Abelianization

Dorigoni-Konishi-Ohashi 2008

- The stability of non-Abelian semi-local vortices

Auzzi-Eto-SBG-Konishi-Vinci 2008

- Multi-layer structure of non-Abelian vortices:

Eto-Fujimori-Nitta-Ohashi-Sakai 2009

- Non-Abelian global vortices:

Eto-Nakano-Nitta 2009

- Vortex description of quantum Hall ferromagnets

Kimura 2009

- Non-Abelian vortex-string dynamics from non-linear realization:

Liu-Nitta 2009

- Non-Abelian vortices in mass-deformed ABJM model:

Kim-Kim-Kwon-Nakajima 2009, Auzzi-Kumar 2009

- Quantum Phases of a vortex string in $\mathcal{N} = 1^{\star}$:

Auzzi-Kumar 2009

- Non-Abelian Chern-Simons vortices with generic gauge groups:

SBG 2009

- Moduli space metric for non-Abelian vortices
  - Compact Riemann surfaces:

    Baptista 2010

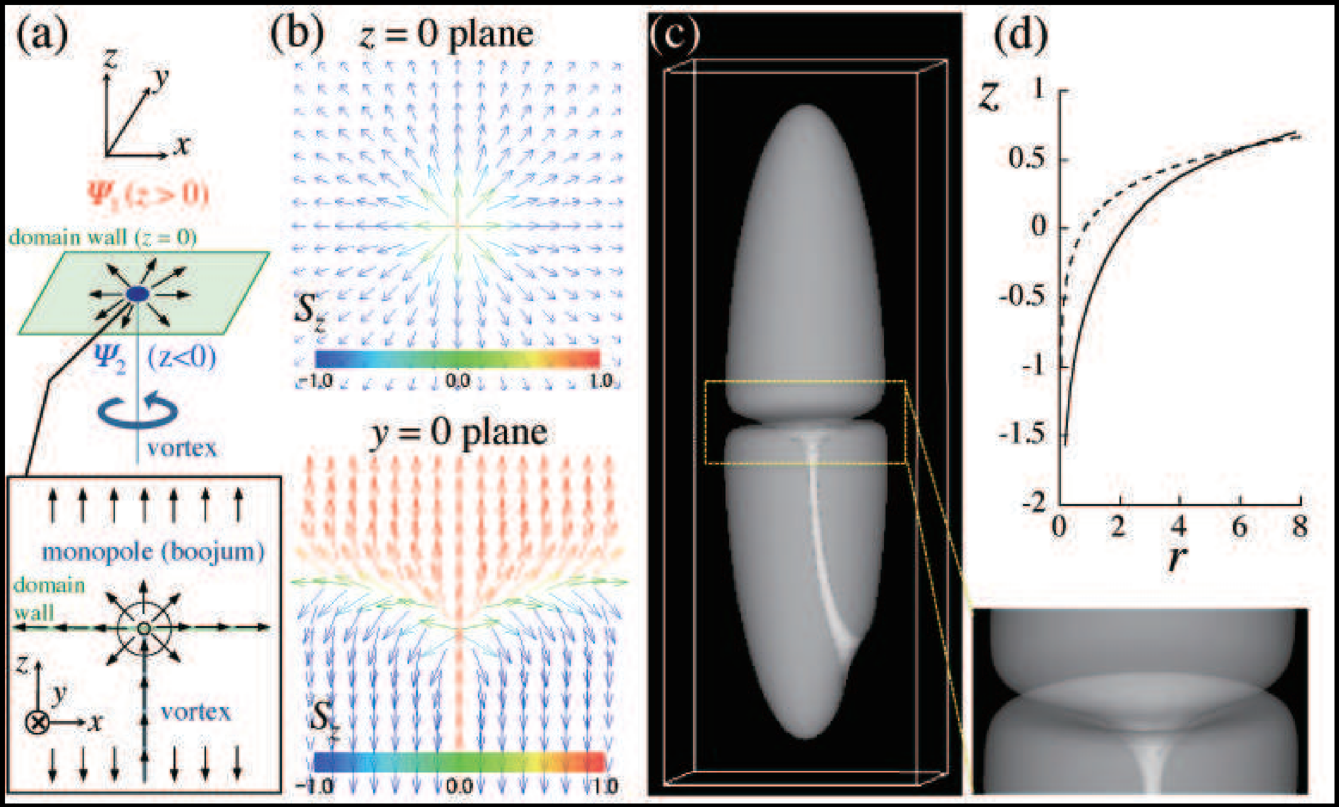
  - Well-separated vortices:

    Fujimori-Marmorini-Nitta-Ohashi-Sakai 2010

- Low-energy $U(1) \times USp(2M)$ gauge theory from simple high-energy gauge group:

SBG-Konishi 2010

- D-branes in rotating phase-separated two-component Bose-Einstein condensates



Kasamatsu-Takeuchi-Nitta-Tsubota 2010

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The model – general gauge group

H theory: $\mathcal{N} = 2$ (8 supercharges), $U(1) \times G'$ super-Yang-Mills (4 dimensions) with N_{F} massless hyper multiplets.

$$\text{vector multiplet :} \quad \{V, \Phi\}$$

$$\text{hyper multiplets :} \quad \{Q, \tilde{Q}\}$$

Truncated model, formally $\mathcal{N} = 1$, $U(1) \times G'$ SYM with N_{F} quarks Q.

$$\mathcal{L} = \mathrm{Tr} \left[\int d^4\theta \left(Q^\dagger e^{-V} Q + \xi V \right) + \frac{1}{g^2} \int d^2\theta \left(WW + \mathrm{h.c.} \right) \right] ,$$

$\xi > 0$ FI parameter.

$$\mathcal{L} = -\frac{1}{4e^2} F^0_{\mu\nu} F^{0,\mu\nu} - \frac{1}{4g^2} F^a_{\mu\nu} F^{a,\mu\nu} + \mathrm{Tr}\, \mathcal{D}_\mu H \left(\mathcal{D}^\mu H \right)^\dagger - \frac{e^2}{2} \left| \mathrm{Tr}\, H H^\dagger t^0 - \frac{v^2}{\sqrt{2N}} \right|^2 - \frac{g^2}{2} \left| \mathrm{Tr}\, H H^\dagger t^a \right|^2 , \quad (13)$$

t^a generators of G'.

BPS equations

Bogomol'nyi bound:

$$T \geq -\frac{v^2}{\sqrt{2N}} \int_{\mathbb{C}} F^0_{12} = 2\pi v^2 \nu \ , \tag{14}$$

saturated by BPS-equations

$$\bar{\mathcal{D}} H = 0 \ , \quad F^0_{12} = e^2 \left(\mathrm{Tr}\, H H^\dagger t^0 - \frac{v^2}{\sqrt{2N}} \right) \ , \tag{15}$$

$$F^a_{12} = g^2 \mathrm{Tr}\, H H^\dagger t^a \ , \tag{16}$$

ν topological charge.

Moduli matrix

Solution: holomorphic matrix + complexified gauge transformations:

$$H = S^{-1}(z, \bar{z})\, H_0(z)\ , \tag{17}$$

with $z = x^1 + ix^2$, holomorphic $N \times N_{\mathrm{F}}$ matrix: moduli matrix, encodes all the moduli.

$$S = sS'\ , \quad s \in \mathbb{C}^\star\ , \quad S' \in {G'}^{\mathbb{C}}\ . \tag{18}$$

$$\bar{\mathcal{D}} H = 0\ , \quad \Rightarrow \quad \bar{A} = -iS^{-1}\bar{\partial} S\ . \tag{19}$$

Residual symmetry – V-equivalence:

$$(S, H_0) \sim V(z)\,(S, H_0)\ . \tag{20}$$

Master equations

$\bar{\mathcal{D}}H = 0$ solved. $\Omega = \omega\Omega' = SS^\dagger$, $\Omega_0 = H_0(z)H_0^\dagger(z)$:
For $SU(N)$

$$\bar\partial\partial\log\omega = -\frac{e^2}{4N}\left[\frac{1}{\omega}\mathrm{Tr}\,\Omega_0\Omega'^{-1} - v^2\right]\ , \tag{21}$$

$$\bar\partial\left(\Omega'\partial\Omega'^{-1}\right) = \frac{g^2}{4\omega}\left[\Omega_0\Omega'^{-1} - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\,\Omega_0\Omega'^{-1}\right]\ , \tag{22}$$

SO, USp:

$$\bar\partial\left(\Omega'\partial\Omega'^{-1}\right) = \frac{g^2}{8\omega}\left[\Omega_0\Omega'^{-1} - J^\dagger\left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}}J\right]\ . \tag{23}$$

Existence and uniqueness is assumed.

Asymptotic behavior:

$$s(z,\bar{z}) \sim |z|^{\nu} \ . \tag{24}$$

Holomorphic G'-invariants:

$$I^{i}_{G'}\left(H = s^{-1}S'^{-1}H_0\right) = s^{-n_i} I^{i}_{G'}(H_0) \ , \tag{25}$$

Boundary conditions:

$$I^{i}_{G'}(H_0)\Big|_{|z|\to\infty} = I^{i}_{\rm vev} z^{\nu n_i} \ , \tag{26}$$

Single valuedness condition

$$\nu n_i \in \mathbb{Z}_+ \ , \tag{27}$$

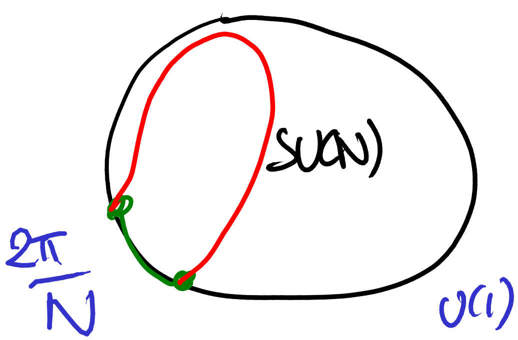
Solution:

$$\nu = \frac{k}{n_0} \ , \qquad k \in \mathbb{Z}_+ \ , \tag{28}$$

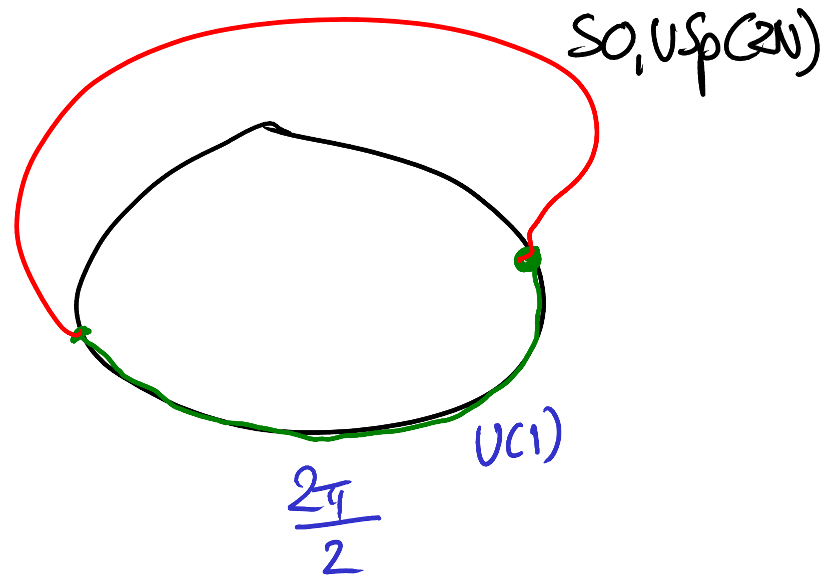
$$\boxed{n_0 \equiv \gcd\left\{n_i | I^{i}_{\rm vev} \neq 0\right\} \ .}$$

$$G = \frac{U(1) \times G'}{\mathbb{Z}_{n_0}} \ . \tag{29}$$

$SU(N) = G'$



$SO, USp(2N) = G'$



A few groups

G	C_G	ν
$SU(N)$	$\mathbb{Z}_N$	k/N
$SO(2N+1)$	$\mathbb{1}$	k
$USp(2N)$	$\mathbb{Z}_2$	$k/2$
$SO(2N)$	$\mathbb{Z}_2$	$k/2$
E_6	$\mathbb{Z}_3$	$k/3$
E_7	$\mathbb{Z}_2$	$k/2$
E_8	$\mathbb{1}$	k
F_4	$\mathbb{1}$	k
G_2	$\mathbb{1}$	k

Examples

Holomorphic invariant conditions:

$G' = SU(N)$:

$$\det H_0(z) = z^k + \mathcal{O}\left(z^{k-1}\right) \ , \tag{30}$$

$G' = SO(2N), USp(2N)$: meson field:

$$M = H^{\mathrm{T}} J H, \tag{31}$$

Invariant tensor:

$$J_{SO(2N)} \equiv \sigma^1 \otimes \mathbf{1}_N \ , \qquad J_{USp(2N)} \equiv i\sigma^2 \otimes \mathbf{1}_N \ , \tag{32}$$

Holomorphic invariant conditions:

$$H_0^{\mathrm{T}} J H_0 = z^k J + \mathcal{O}\left(z^{k-1}\right) \ . \tag{33}$$

$G' = SO(2N+1)$:

$$H_0^{\mathrm{T}} J H_0 = z^{2k} J + \mathcal{O}\left(z^{2k-1}\right) \ . \tag{34}$$

Special points

$$H_0(z) = z^{\nu \mathbf{1}_N + \nu_\alpha \mathcal{H}_\alpha} \in U(1)^{\mathbb{C}} \times {G'}^{\mathbb{C}} \ , \qquad (35)$$

$\mathcal{H}_\alpha$: Cartan generators of $\mathfrak{g}'$, $\alpha = 1, \ldots, \mathrm{rank}(G')$

$$\boxed{\nu = \frac{k}{n_0} \ .}$$

Single valued condition:

$$(\nu \mathbf{1}_N + \nu_\alpha \mathcal{H}_\alpha)_{ii} \in \mathbb{Z}_{\geq 0} \quad \forall i \ , \qquad \Rightarrow \nu + \nu_\alpha \mu_\alpha^{(j)} \in \mathbb{Z}_{\geq 0} \quad \forall i \ ,$$

$\mu_\alpha^{(j)}$: weight vector of G'.

Quantization with respect to the dual group

Quantization condition:

$$\boxed{\vec{\nu} \cdot \vec{\alpha}^{(j)} \in \mathbb{Z} \ ,}$$

$\alpha^{(j)}$: root vectors of G'.

Formally Goddard-Nuyts-Olive-Weinberg quantization condition.

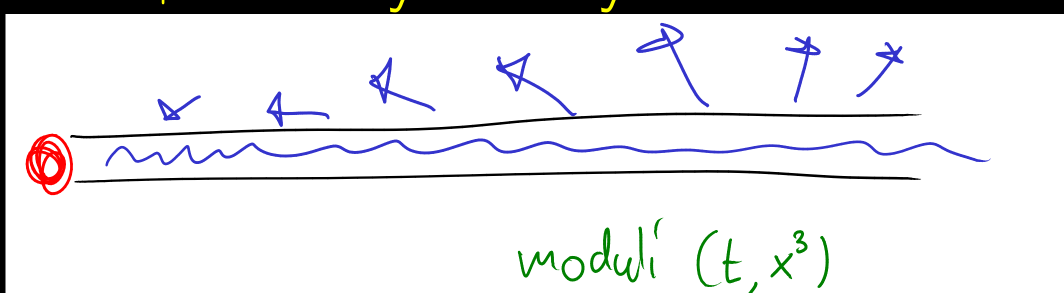
Solution:

$$\vec{\nu} = 2\vec{\mu} \ , \tag{36}$$

$\vec{\mu}$: any weight vector of dual group, dual group's root vectors

$$\vec{\alpha}^* = \frac{\vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} \ . \tag{37}$$

No conceptual problems in quantizing zero-modes due to exact color+flavor symmetry.

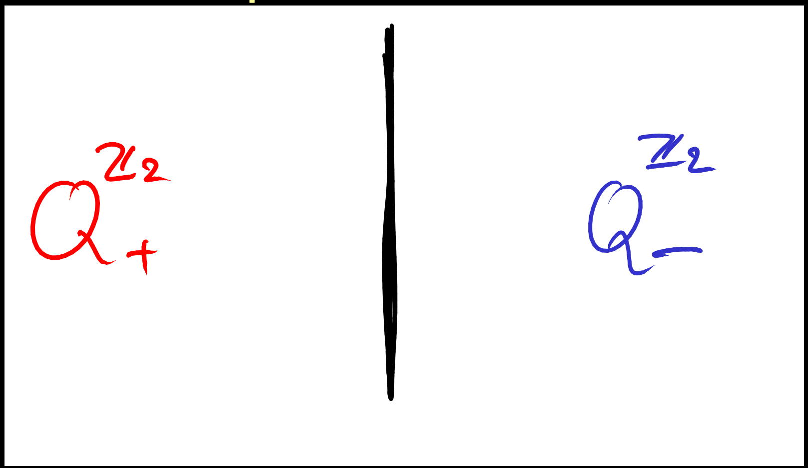


$\mathbb{Z}_2$ parity for $G' = SO(N)$

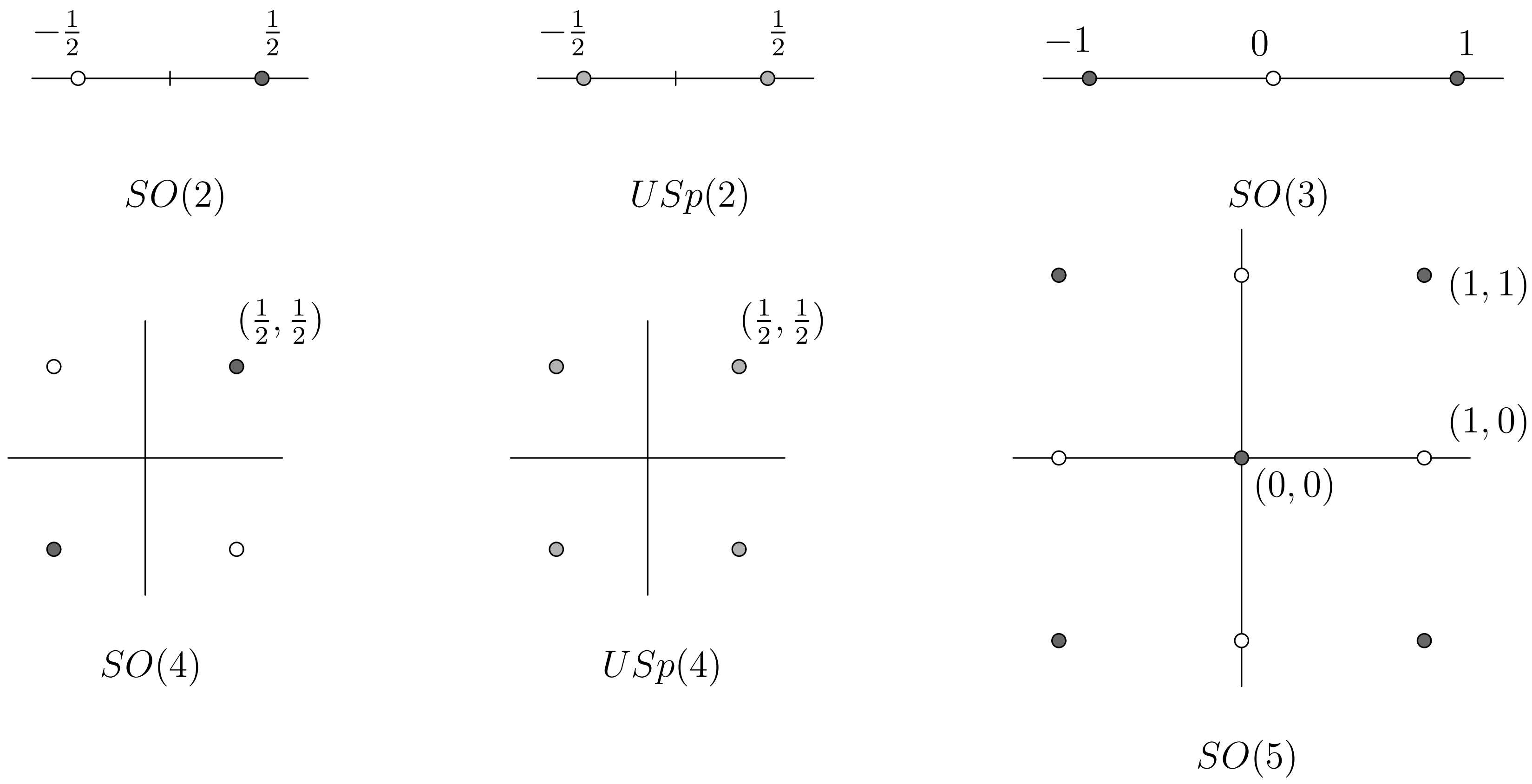
First homotopy group

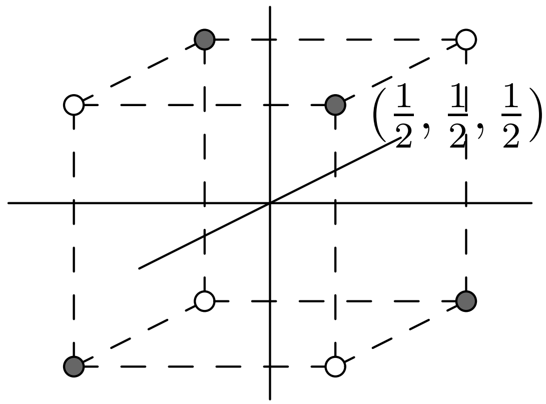
$$\pi_1\left(\frac{U(1)\times SO(N)}{Z_{n_0}}\right) = \mathbb{Z}\times\mathbb{Z}_2 \ , \qquad \begin{cases} n_0 = 1 \ , & N \text{ odd} \\ n_0 = 2 \ , & N \text{ even} \end{cases} \tag{38}$$

Moduli space is disconnected:

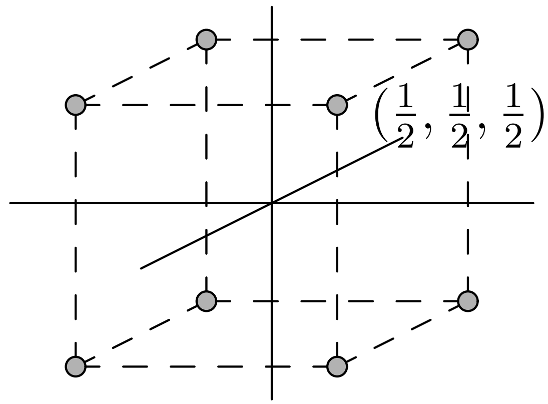


Special points for $k = 1$ vortex with $G' = SO, USp$.





$SO(6)$



$USp(6)$

Dark grey points have positive $\mathbb{Z}_2$-charge, white ones have negative.

Moduli space of vortices

The moduli space of vortices: completely described by moduli matrix

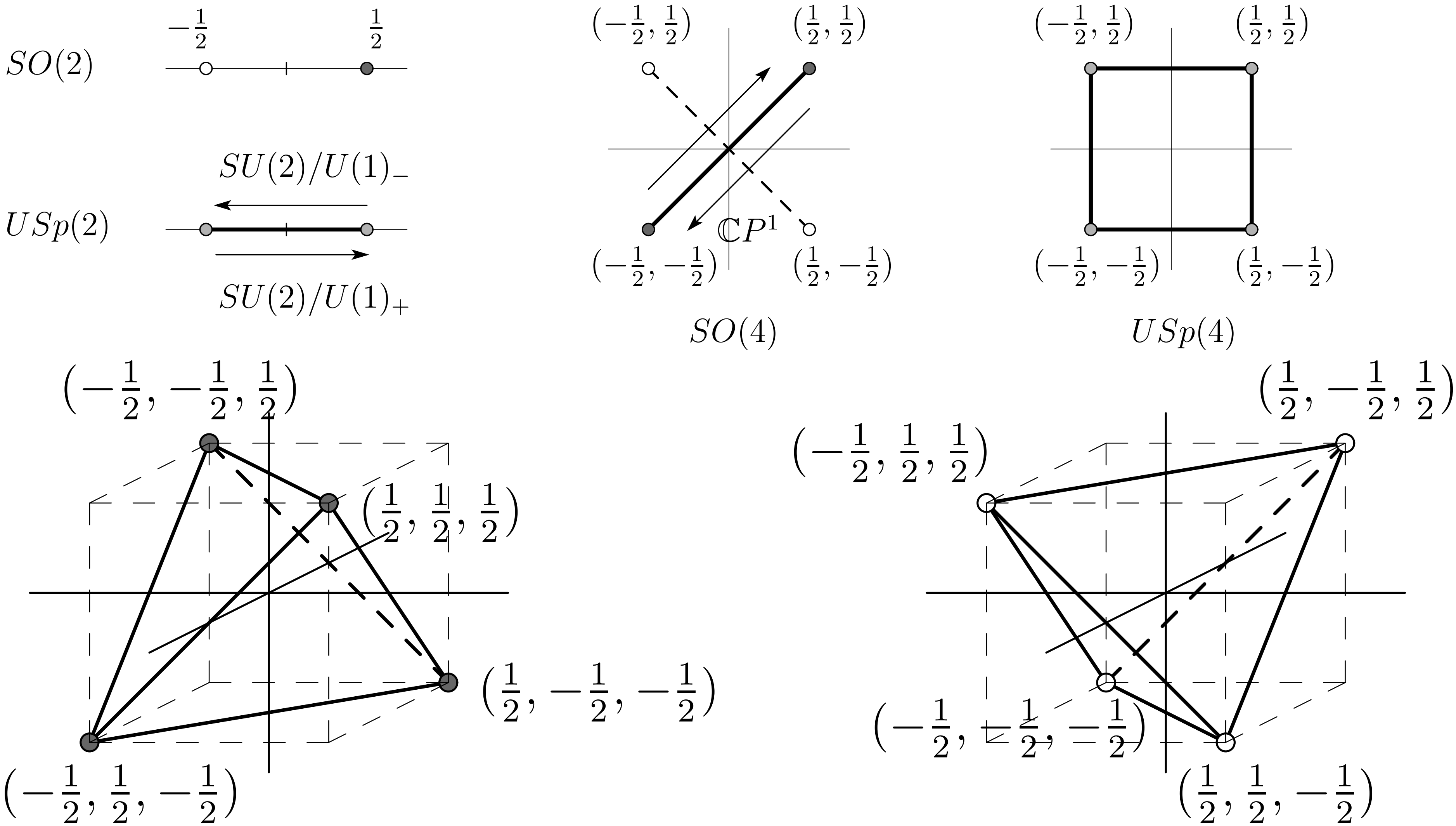
$$\mathcal{M} = \{H_0(z)\} /\!\!/ U(N)^{\mathbb{C}} \ . \tag{39}$$

Some examples: $k = 1$

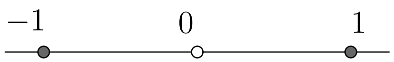
$$\mathcal{M}_{USp(2M)} = \mathbb{C} \times \frac{USp(2M)}{U(M)} \ , \tag{40}$$

$$\mathcal{M}_{SO(2M)} = \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_+ \cup \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_- \ , \tag{41}$$

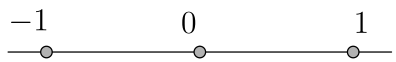
Connectedness of the moduli space



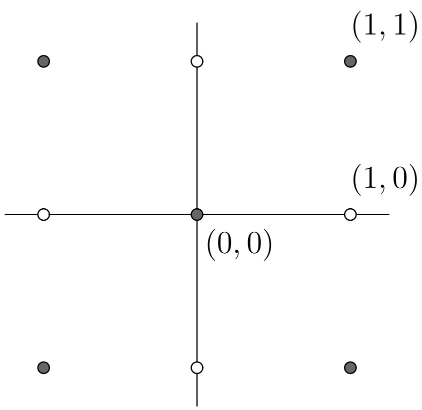
The $k = 1$ odd SO "⇔" $k = 2$ even:



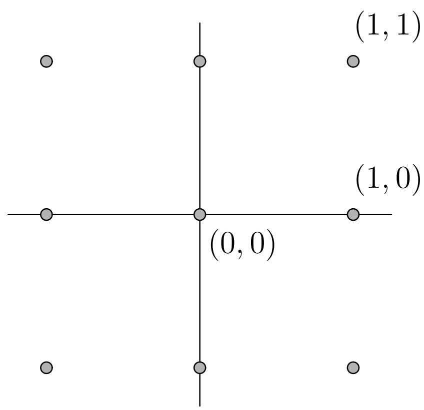
$SO(2)$



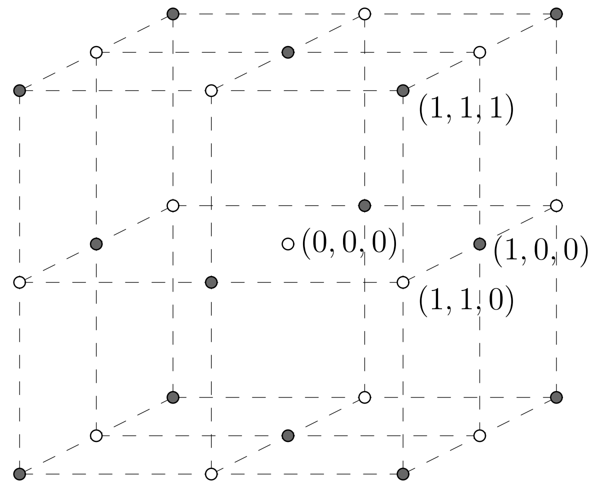
$USp(2)$



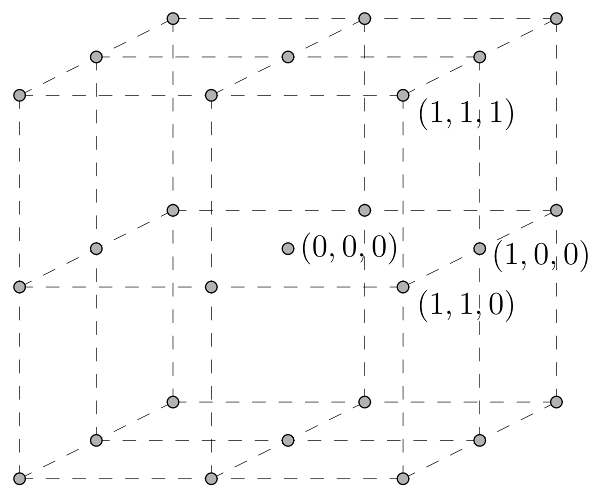
$SO(4)$



$USp(4)$

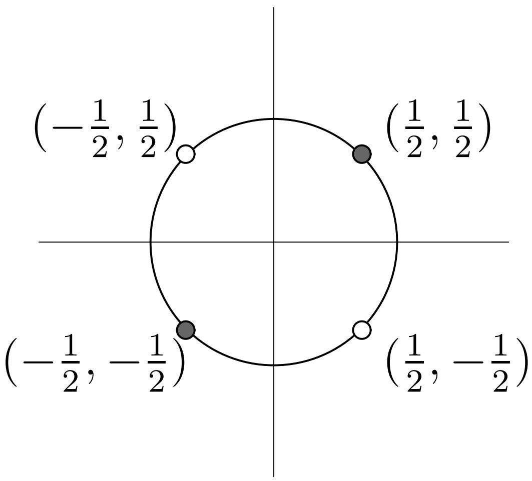


$SO(6)$

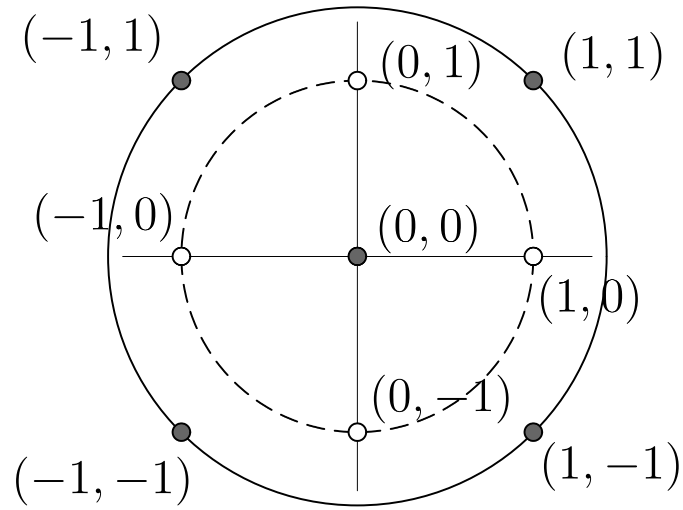


$USp(6)$

Differences

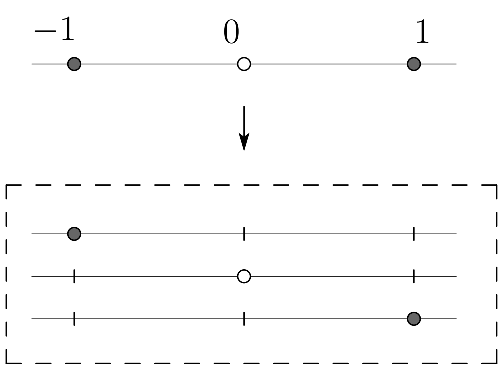


$SO(4)$

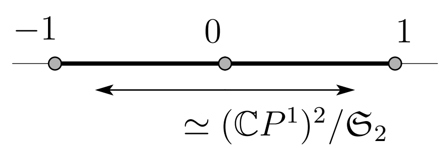


$SO(5)$

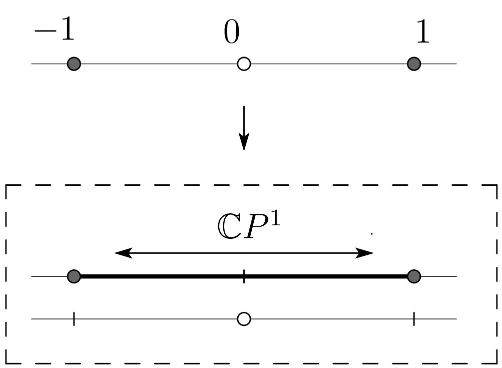
$k = 2,\ SO(2)$



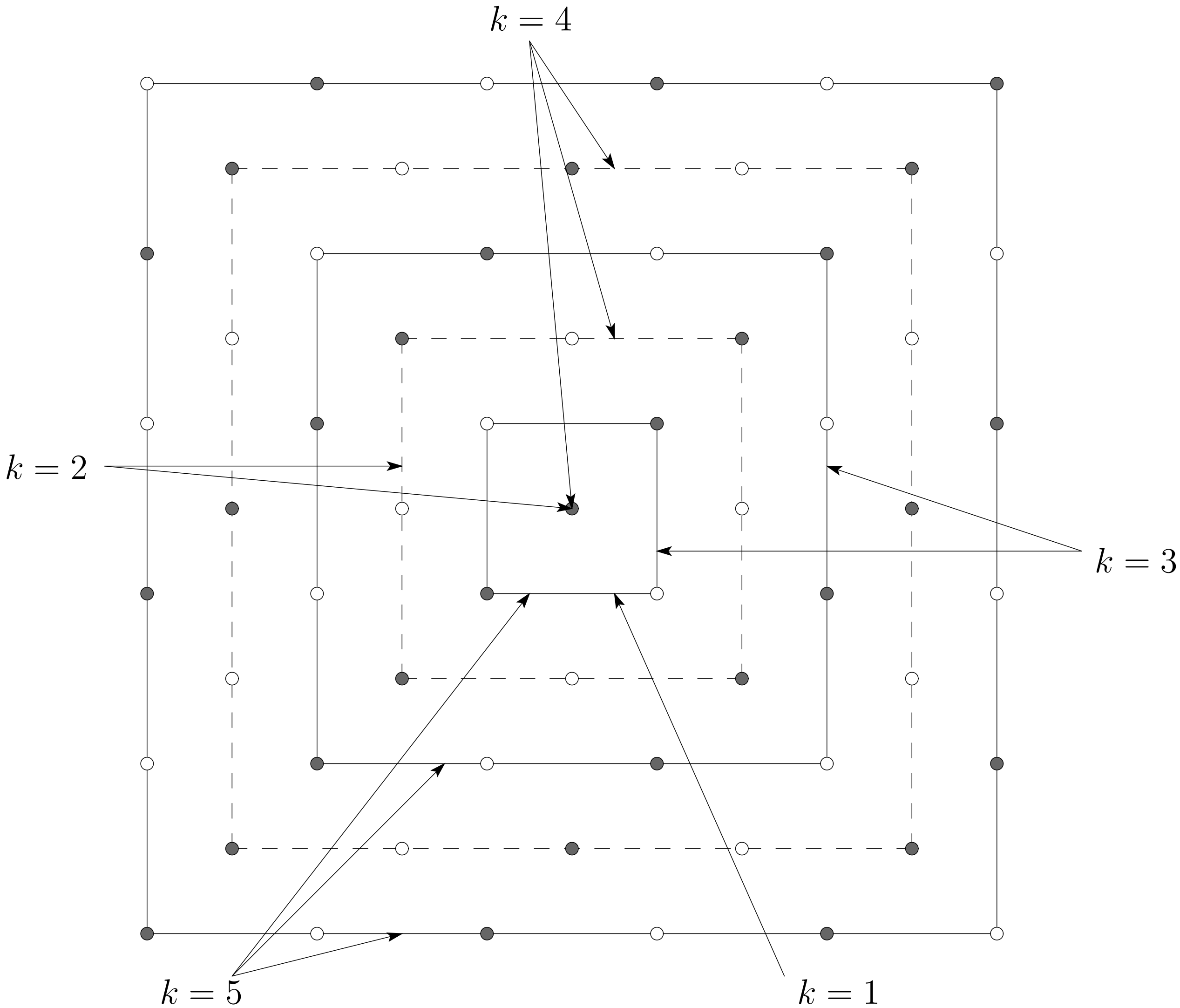
$k = 2,\ USp(2)$



$k = 1,\ SO(3)$



$SO(4)$ higher windings



Plan of the talk

Introductory part

- motivation
- crash course on vortices
- summary of recent results in the field

Results

- SO, USp theories

$\longrightarrow$ ◦ NLσM lumps

- fractional vortices
- Chern-Simons vortices

Existence and uniqueness

1. Result of our index calculation with generic gauge group

   $$\# \text{ bosonic zero-modes} \Leftrightarrow \# \text{ moduli in } H_0(z)$$

2. Strong gauge coupling limit $\Rightarrow$ non-linear σ-model
   analytic solution is obtained

Strong gauge coupling limit

$SU(N)$:

$$0 = \frac{1}{\omega} \mathrm{Tr}\, \Omega_0 {\Omega'}^{-1} - v^2 \ , \tag{42}$$

$$0 = \Omega_0 {\Omega'}^{-1} - \frac{\mathbf{1}_N}{N} \mathrm{Tr}\, \Omega_0 {\Omega'}^{-1} \ , \tag{43}$$

SO, USp:

$$0 = \Omega_0 {\Omega'}^{-1} - J^\dagger \left(\Omega_0 {\Omega'}^{-1} \right)^{\mathrm{T}} J \ , \tag{44}$$

Solution for $SU(N)$:

$$\omega_\infty = \frac{N}{v^2} (\det \Omega_0)^{\frac{1}{N}} \ , \qquad \Omega'_\infty = (\det \Omega_0)^{-\frac{1}{N}} \Omega_0 \ , \tag{45}$$

Solution for SO, USp: $(M \equiv H_0^{\mathrm{T}}(z) J H_0)$

$$\omega_\infty = \frac{1}{v^2} \mathrm{Tr} \sqrt{M^\dagger M} \ , \qquad \Omega'_\infty = H_0(z) \frac{\mathbf{1}_N}{\sqrt{M^\dagger M}} H_0^\dagger(z) \ . \tag{46}$$

Lumps

Low energy		High energy
NLσM	$\Leftrightarrow$	YM-Higgs
stringy lumps		semi-local vortices
$H_0(z)$		$H_0(z)$
target space		Higgs branch

The NLσM is integrable.

Except for the local vortex which is mapped to a point
$\Rightarrow$ small lump singularity.

Kähler quotient

$U(N)$:

$$\mathcal{L} = \mathrm{Tr} \int d^4\theta \ \{QQ^\dagger e^{-V} + \xi V\} \ , \tag{47}$$

$$\sim \int d^4\theta \ \xi \log\det QQ^\dagger \ , \tag{48}$$

SO, USp:

$$\mathcal{L} = \mathrm{Tr} \int d^4\theta \ \left\{QQ^\dagger e^{-V'} e^{-V_e} + \xi V_e\right\} \ , \tag{49}$$

$e^{-V'} \in SO, USp$.

Difficult calculation.

Kähler quotient for SO, USp

Relax the algebra $e^{-V'} \in SL(N, \mathbb{C})$, introduce Lagrange multipliers λ.

$$\mathcal{L} = \mathrm{Tr} \int d^4\theta \left\{ QQ^\dagger e^{-V'} e^{-V_e} + \lambda \left(e^{-V'^{\mathrm{T}}} J e^{-V'} - J \right) + \xi V_e \right\} ,$$

$$\sim \int d^4\theta \; \xi \log \mathrm{Tr} \sqrt{MM^\dagger} \, , \qquad (50)$$

$M = Q^{\mathrm{T}} J Q$ is the meson field.

A similar construction has been made for the hyper-Kähler case ($\mathcal{N} = 2$).

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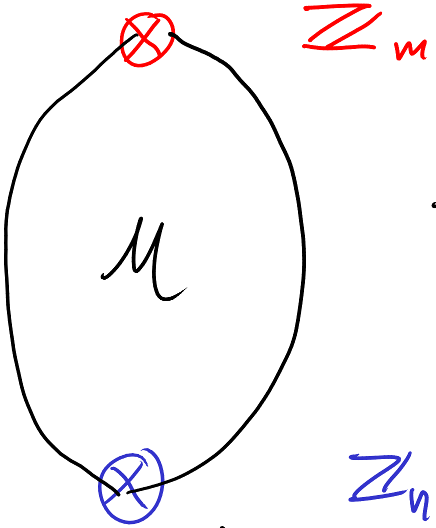
Fractional vortices of the first type

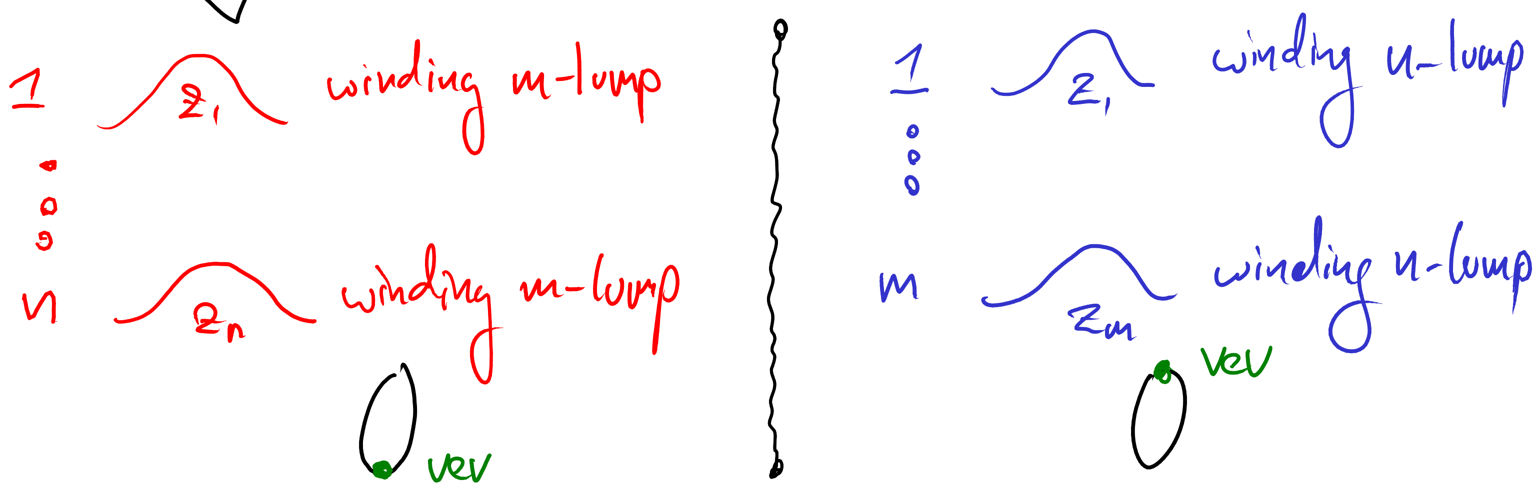
$\mathcal{M} \sim S^2$ with two singularities

m, n rel. prime



Now to avoid a singular field configuration the 2-cycle should wrap: m × n times

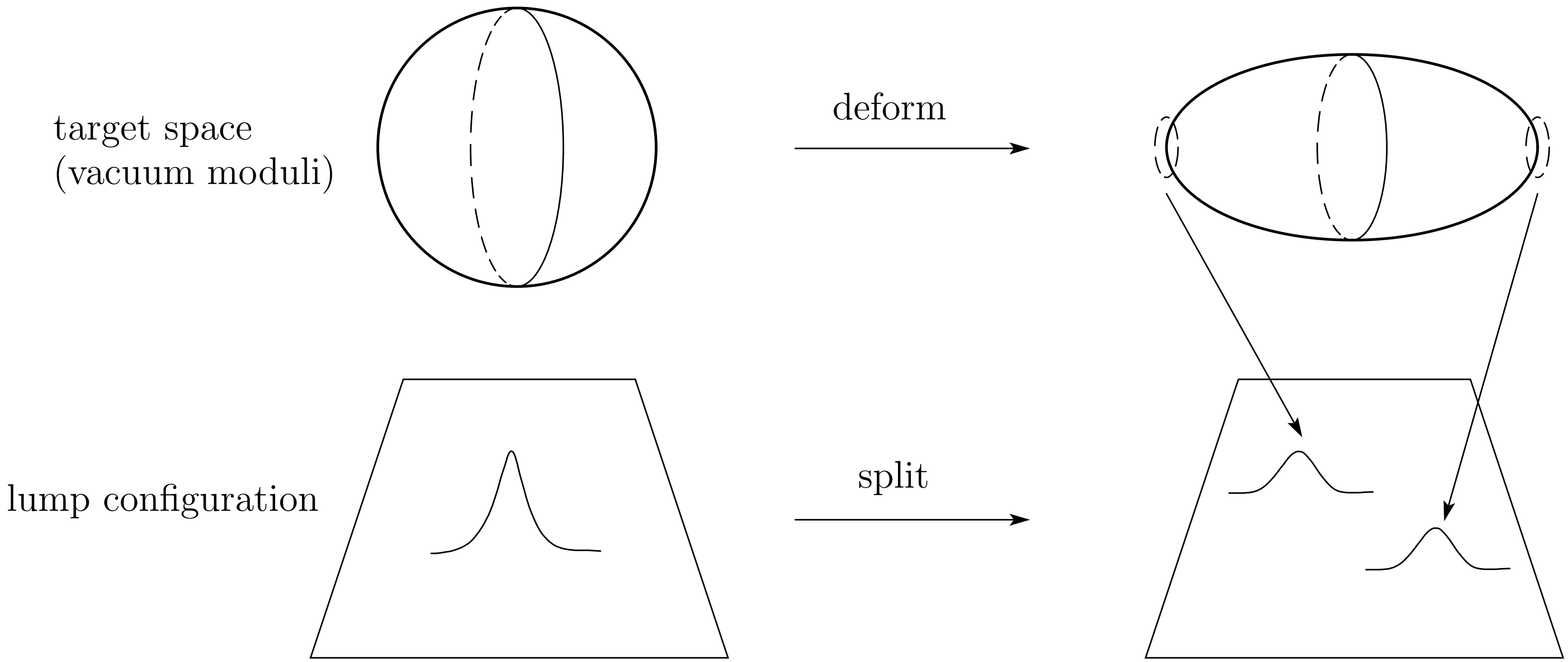
$\Downarrow$



Formally we can write

$$R(z) = R_{rev} \frac{\prod_{j=1}^{m} (z - z_j)^{n}}{\prod_{i=1}^{n} (z - z_i)^{m}}$$

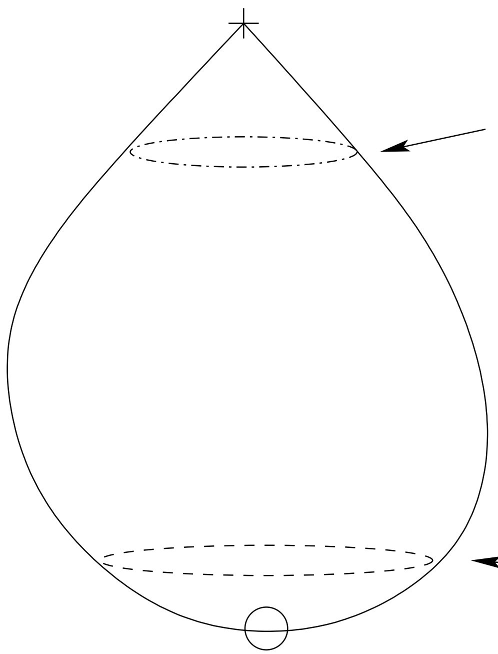
Fractional vortices of the second type



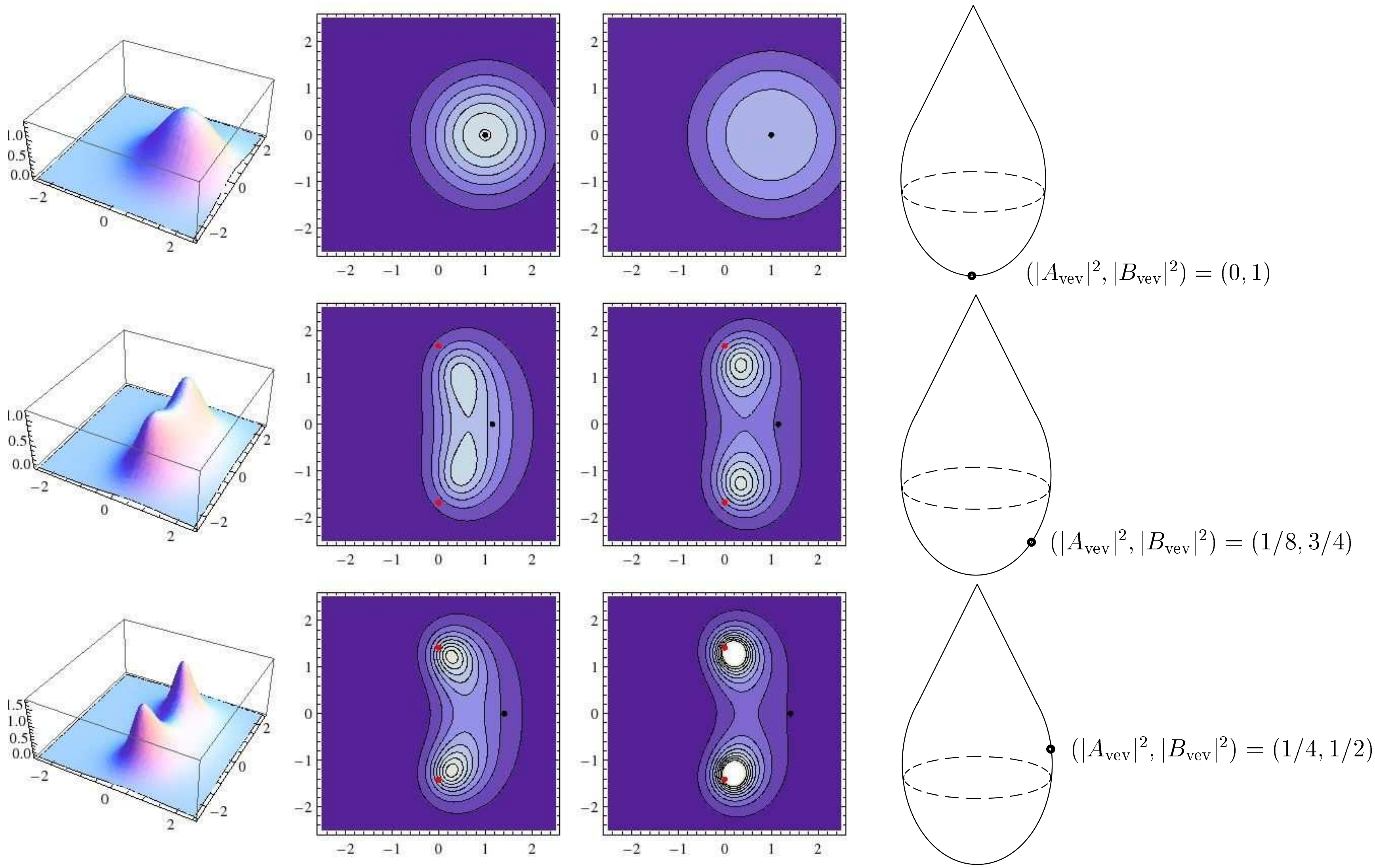
The droplet model

	$U(1)$
A	2
B	1

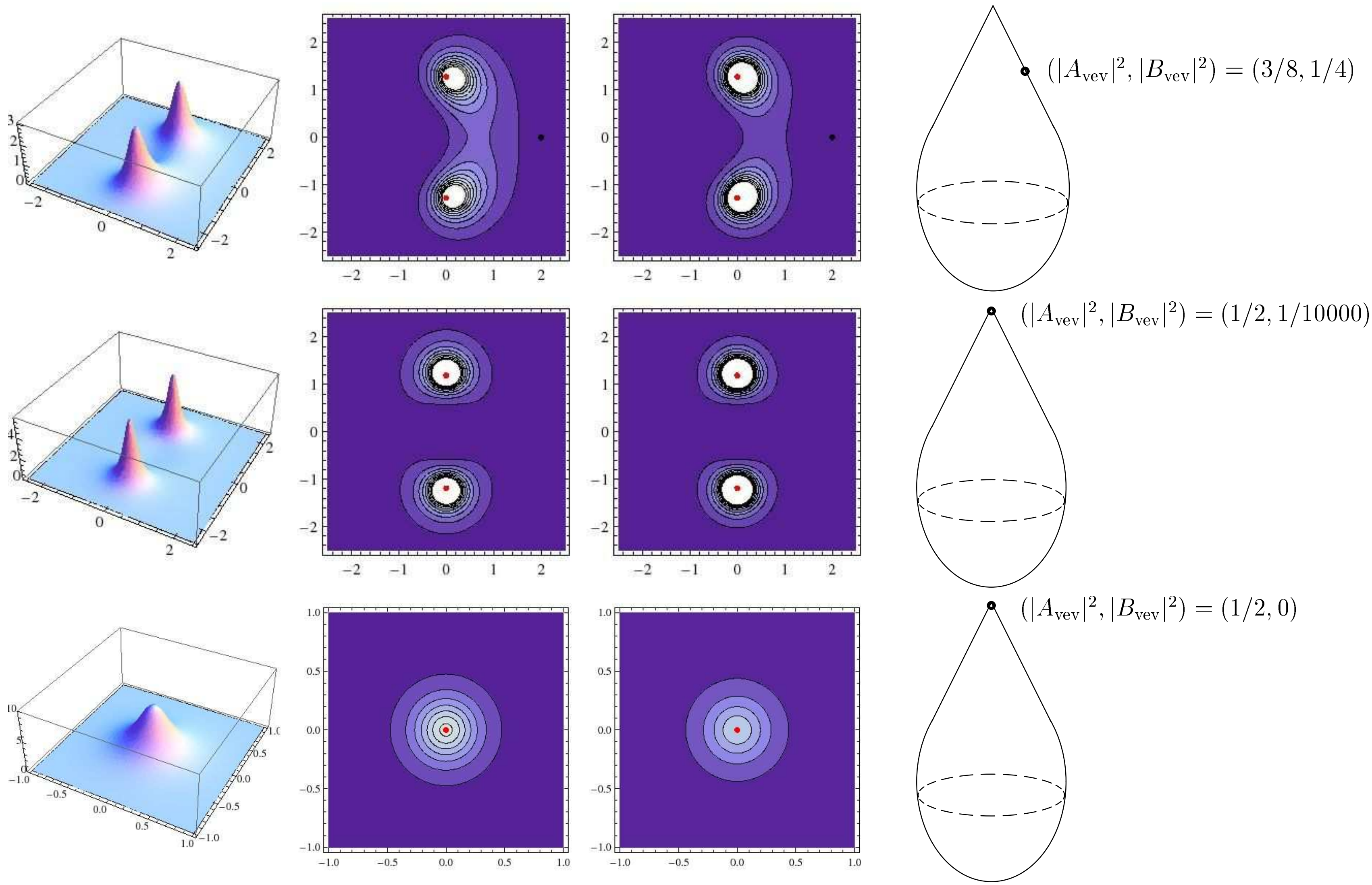
$$H = \begin{pmatrix} A \\ B \end{pmatrix} . \qquad (51)$$



$$\phi(z) = \phi_{\rm vev} \frac{(z - z_1^{\rm S})(z - z_2^{\rm S})}{(z - z_1^{\rm N})^2} . \qquad (52)$$

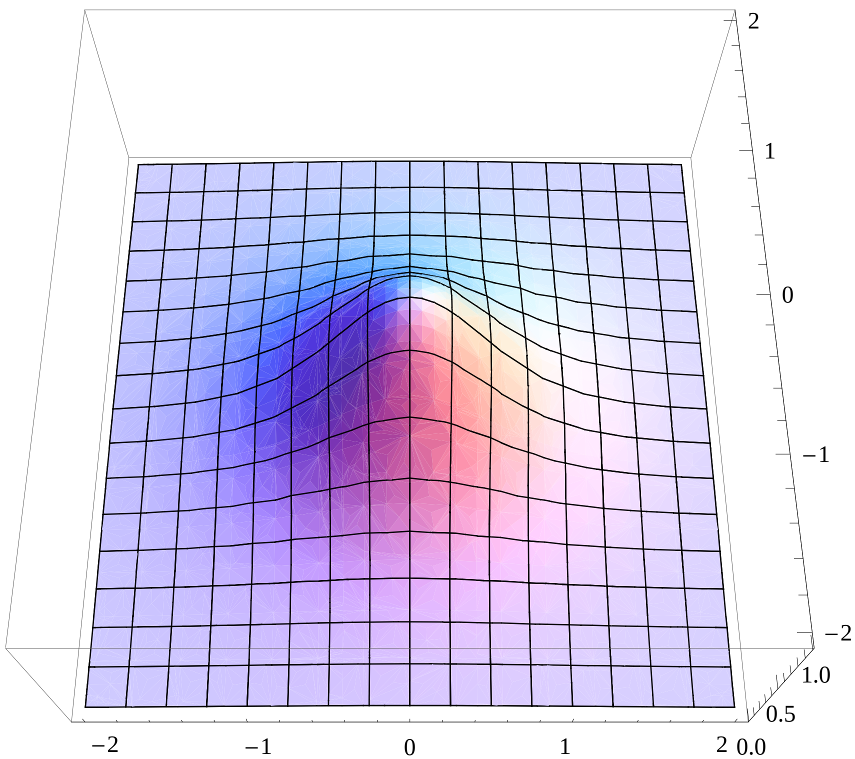


$(|A_{\rm vev}|^2, |B_{\rm vev}|^2) = (0, 1)$
$(|A_{\rm vev}|^2, |B_{\rm vev}|^2) = (1/8, 3/4)$
$(|A_{\rm vev}|^2, |B_{\rm vev}|^2) = (1/4, 1/2)$



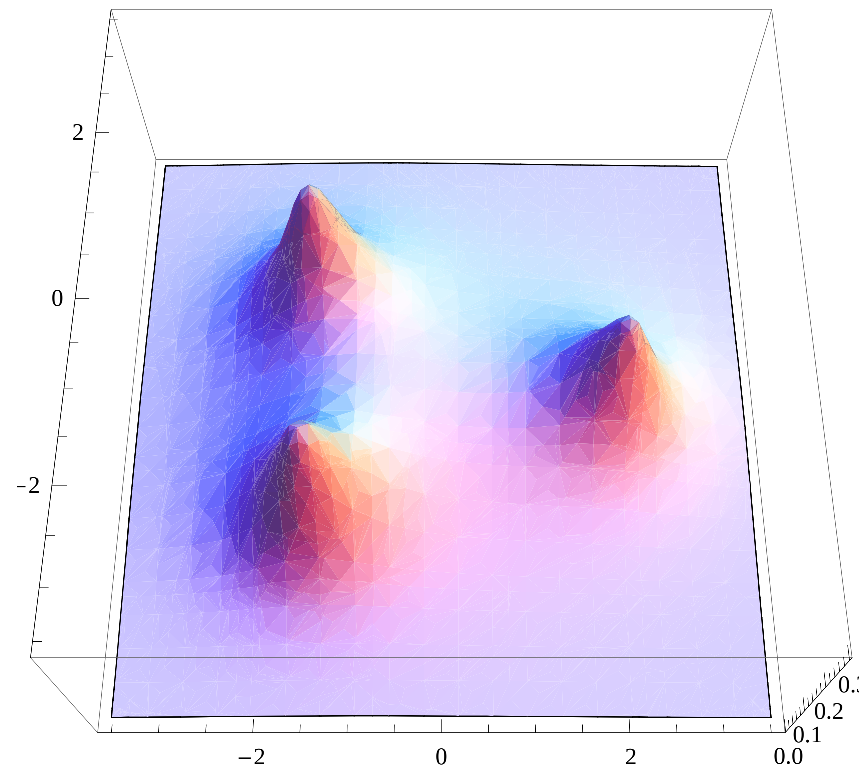
$(|A_{\rm vev}|^2, |B_{\rm vev}|^2) = (3/8, 1/4)$
$(|A_{\rm vev}|^2, |B_{\rm vev}|^2) = (1/2, 1/10000)$
$(|A_{\rm vev}|^2, |B_{\rm vev}|^2) = (1/2, 0)$

Example: $SO(6)$ theory



$$1$$

$\Rightarrow$



$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

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$\longrightarrow$ ○ Chern-Simons vortices

Yang-Mills-Chern-Simons-Higgs theory

$\mathcal{N} = 2$ (4 supercharges) in $d = 2+1$ dimensions with the gauge group $G = U(1) \times G'$, where G' is a simple group.

$$\begin{aligned}
\mathcal{L}_{\mathrm{YMCSH}} = &- \frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{4e^2}\left(F^0_{\mu\nu}\right)^2 - \frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho} A^0_\mu \partial_\nu A^0_\rho \\
&- \frac{\mu}{8\pi}\epsilon^{\mu\nu\rho}\left(A^a_\mu \partial_\nu A^a_\rho - \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\right) \\
&+ \frac{1}{2g^2}\left(\mathcal{D}_\mu \phi^a\right)^2 + \frac{1}{2e^2}\left(\partial_\mu \phi^0\right)^2 + \mathrm{Tr}\,\left(\mathcal{D}_\mu H\right)\left(\mathcal{D}^\mu H\right)^\dagger \\
&- \mathrm{Tr}\,\left|\phi H - H m\right|^2 - \frac{g^2}{2}\left(\mathrm{Tr}\,\left(H H^\dagger t^a\right) - \frac{\mu}{4\pi}\phi^a\right)^2 \\
&- \frac{e^2}{2}\left(\mathrm{Tr}\,\left(H H^\dagger t^0\right) - \frac{\kappa}{4\pi}\phi^0 - \frac{1}{\sqrt{2N}}\xi\right)^2 ,
\end{aligned}$$

Integrate out the adjoint fields

Strong gauge coupling limit $e, g \to \infty$:

$$\phi^a = \frac{4\pi}{\mu} \mathrm{Tr}\, \left(H H^\dagger t^a \right) \ , \tag{53}$$

$$\phi^0 = \frac{4\pi}{\kappa} \frac{1}{\sqrt{2N}} \left[\mathrm{Tr}\, \left(H H^\dagger \right) - \xi \right] \ . \tag{54}$$

Chern-Simons-Higgs theory

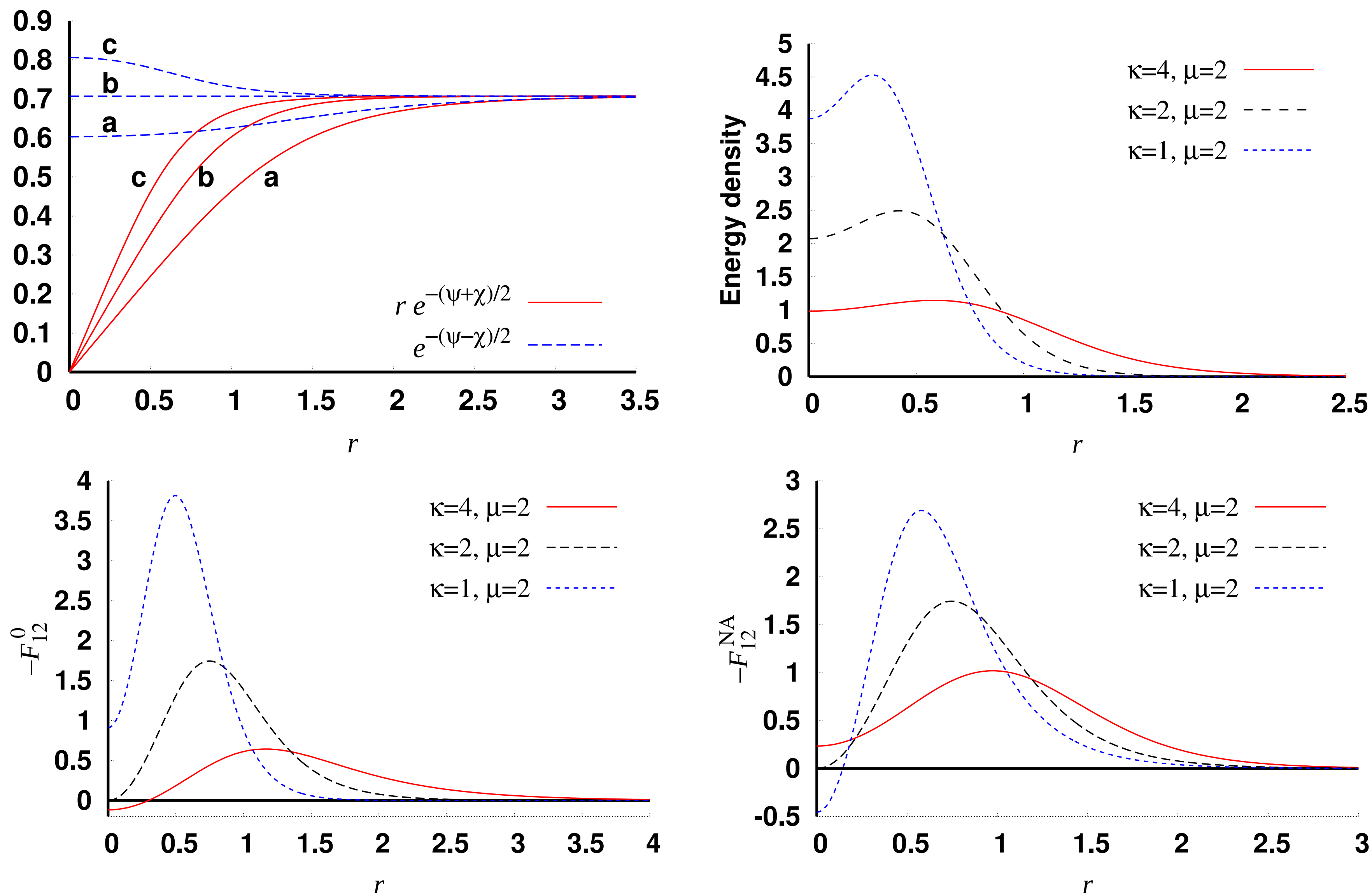
$$\mathcal{L}_{\mathrm{CSH}} = -\frac{\mu}{8\pi}\epsilon^{\mu\nu\rho}\left(A^a_\mu\partial_\nu A^a_\rho - \frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right) - \frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}\left(A^0_\mu\partial_\nu A^0_\rho\right)$$
$$+ \mathrm{Tr}\,\left(\mathcal{D}_\mu H\right)^\dagger\left(\mathcal{D}^\mu H\right)$$
$$- 4\pi^2\mathrm{Tr}\,\left|\left\{\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\,\left(HH^\dagger\right) - \xi\right) + \frac{2}{\mu}\mathrm{Tr}\,\left(HH^\dagger t^a\right)t^a\right\}H\right|^2 , \quad (55)$$

Master equations

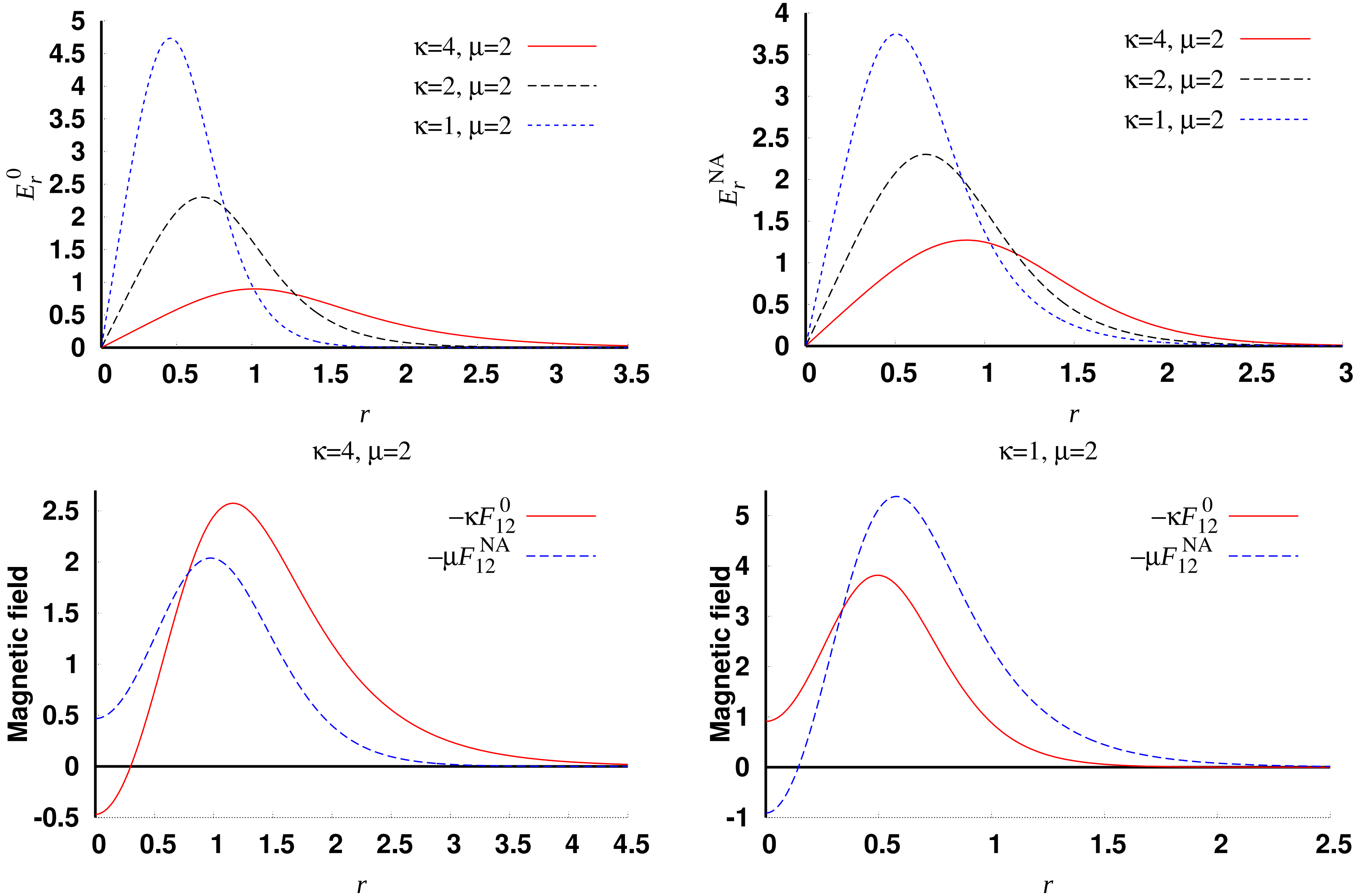
$$\bar{\partial}\left[\Omega'\partial\Omega'^{-1}\right] = \frac{2\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\ \left(\Omega_0\Omega^{-1}\right) - \xi\right)\left\langle\Omega_0\Omega^{-1}\right\rangle_J + \frac{\pi^2}{\mu^2}\left\langle\left(\Omega_0\Omega^{-1}\right)^2\right\rangle_J\ , \qquad (56)$$

$$\bar{\partial}\partial\log\omega = -\frac{4\pi^2}{N^2\kappa^2}\mathrm{Tr}\ \left(\Omega_0\Omega^{-1}\right)\left(\mathrm{Tr}\ \left(\Omega_0\Omega^{-1}\right) - \xi\right) - \frac{2\pi^2}{N\kappa\mu}\mathrm{Tr}\ \left(\Omega_0\Omega^{-1}\left\langle\Omega_0\Omega^{-1}\right\rangle_J\right)\ . \qquad (57)$$

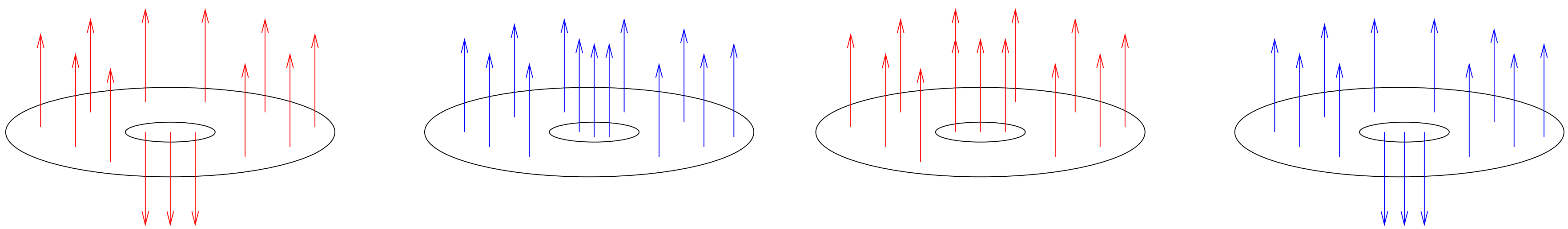
$$\langle X\rangle_J \equiv X - J^\dagger X^{\mathrm{T}} J\ .$$



0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
c
b
a
c
b
a
$r\, e^{-(\psi+\chi)/2}$
$e^{-(\psi-\chi)/2}$
0
0.5
1
1.5
2
2.5
3
3.5
r
Energy density
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
κ=4, μ=2
κ=2, μ=2
κ=1, μ=2
0
0.5
1
1.5
2
2.5
r
$-F^{0}_{12}$
4
3.5
3
2.5
2
1.5
1
0.5
0
κ=4, μ=2
κ=2, μ=2
κ=1, μ=2
0
0.5
1
1.5
2
2.5
3
3.5
4
r
$-F^{\rm NA}_{12}$
3
2.5
2
1.5
1
0.5
0
-0.5
κ=4, μ=2
κ=2, μ=2
κ=1, μ=2
0
0.5
1
1.5
2
2.5
3
r

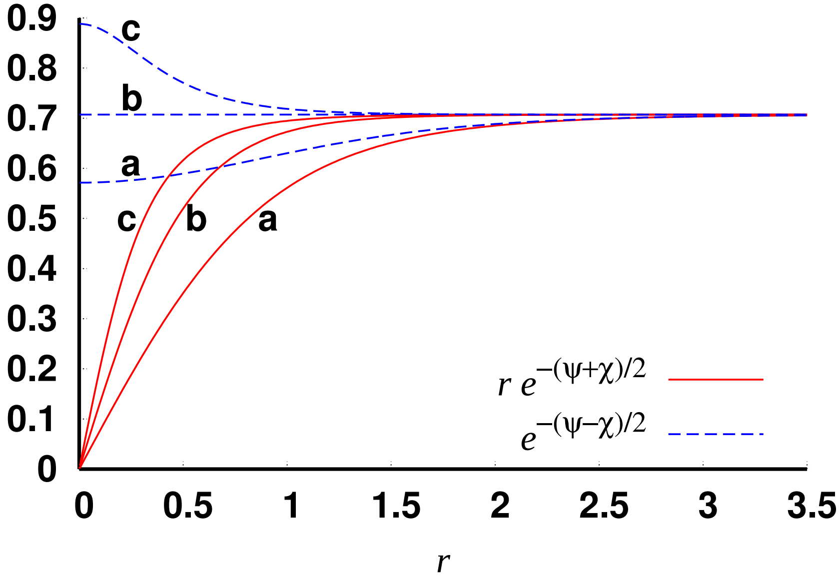


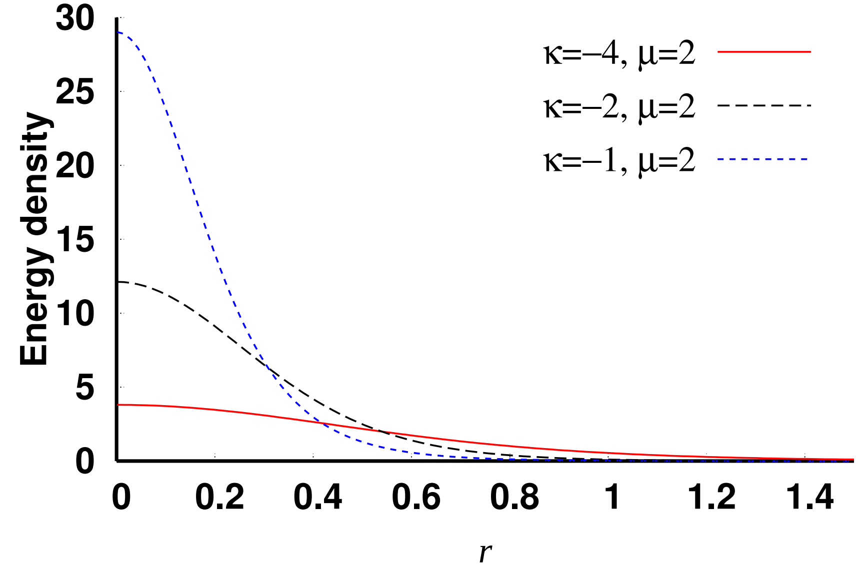
κ=4, μ=2
κ=2, μ=2
κ=1, μ=2
E_r^0
r
κ=4, μ=2
κ=2, μ=2
κ=1, μ=2
E_r^{NA}
r
κ=4, μ=2
$-\kappa F_{12}^0$
$-\mu F_{12}^{NA}$
Magnetic field
r
κ=1, μ=2
$-\kappa F_{12}^0$
$-\mu F_{12}^{NA}$
Magnetic field
r

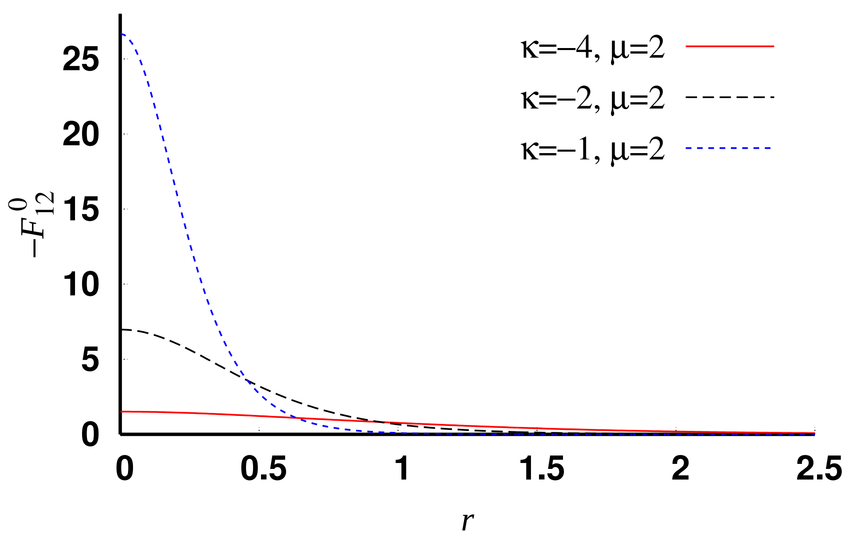


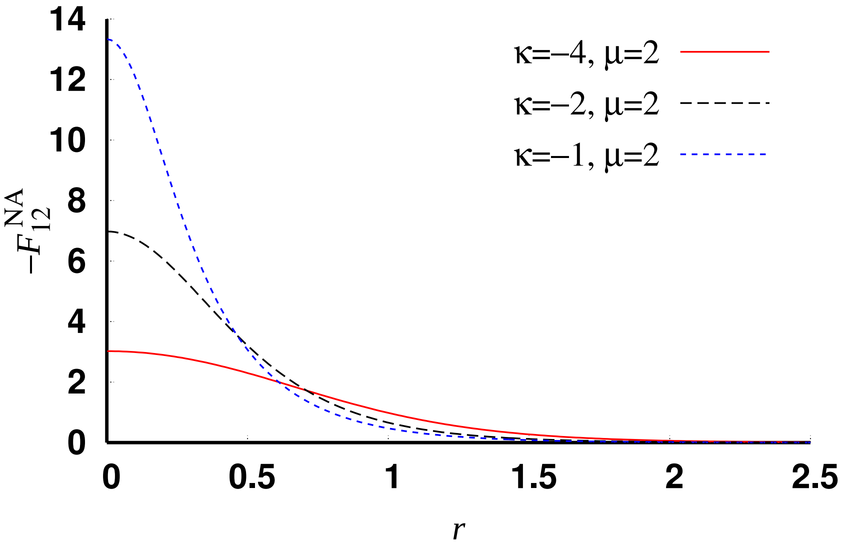
Abelian magnetic field Non−Abelian magnetic field Abelian magnetic field Non−Abelian magnetic field

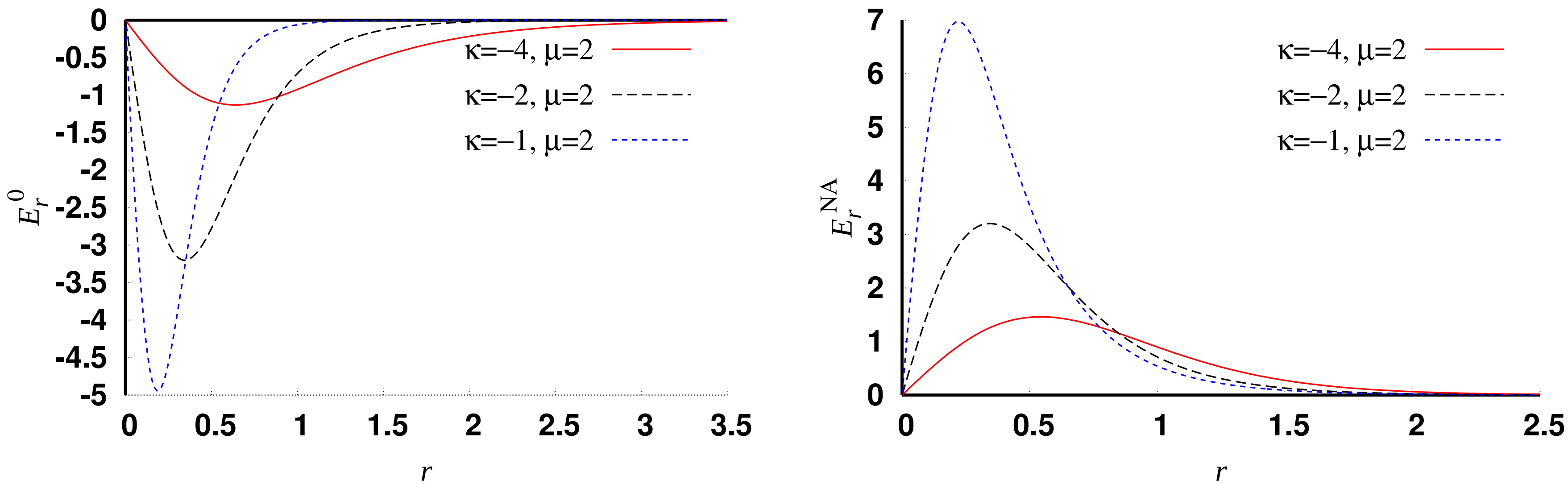
Opposite sign of couplings











E_r^0
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
0
0.5
1
1.5
2
2.5
3
3.5
r
κ=−4, μ=2
κ=−2, μ=2
κ=−1, μ=2
E_r^{NA}
7
6
5
4
3
2
1
0
0
0.5
1
1.5
2
2.5
r
κ=−4, μ=2
κ=−2, μ=2
κ=−1, μ=2

Fractional Chern-Simons vortex

$SO(2M)$:

$$H_0 = \left(\begin{array}{ccc|ccc} z-z_1 & & & c_1 & & \\ & \ddots & & & \ddots & \\ & & z-z_M & & & c_M \\ \hline 0 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 1 \end{array}\right) , \quad (58)$$

$$\Omega' = \mathrm{diag}\left(e^{\chi_1}, \ldots, e^{\chi_M}, e^{-\chi_1}, \ldots, e^{-\chi_M}\right) , \quad (59)$$

$$\omega = e^{\psi} . \quad (60)$$

Asymptotic profiles

$$\delta\chi_m = \frac{2|c_m|^2}{m_\mu^2}|z|^{-4}$$

$$+ \frac{2|c_m|^2}{m_\mu^2}\left[3\left(\frac{z_m}{z}+\frac{\bar{z}_m}{\bar{z}}\right) - \frac{1}{M}\sum_{n=1}^{M}\left(\frac{z_n}{z}+\frac{\bar{z}_n}{\bar{z}}\right)\right]|z|^{-4} + \mathcal{O}\left(|z|^{-6}\right)$$

$$\delta\psi = \frac{1}{M m_\kappa^2}\left(\sum_{n=1}^{M}\left(|z_n|^2+2|c_n|^2\right) - \frac{1}{M}\left|\sum_{n=1}^{M} z_n\right|^2\right)|z|^{-4}$$

$$+ \frac{1}{M m_\kappa^2}\left[\frac{1}{2}\sum_{n=1}^{M}\left(|z_n|^2+4|c_n|^2\right)\left(\frac{z_n}{z}+\frac{\bar{z}_n}{\bar{z}}\right)\right.$$

$$- \frac{1}{2M}\left(\sum_{n=1}^{M}\frac{z_n^2}{z}\sum_{n'=1}^{M}\bar{z}_{n'} + \sum_{n=1}^{M} z_n \sum_{n'=1}^{M}\frac{\bar{z}_{n'}^2}{\bar{z}}\right)$$

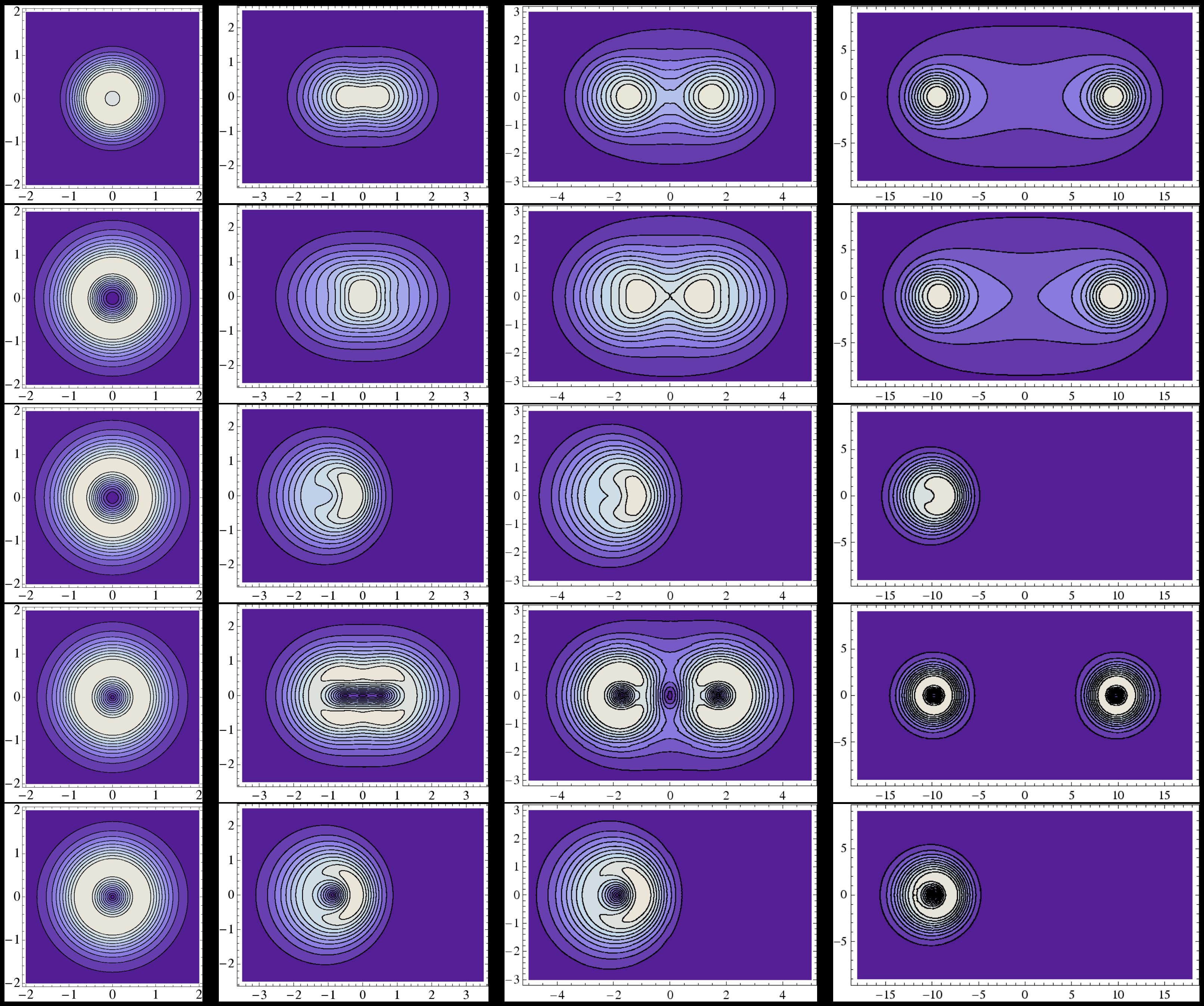
$$\left. + \frac{1}{M}\left(\sum_{n=1}^{M}\left(|z_n|^2+2|c_n|^2\right) - \frac{1}{M}\left|\sum_{n=1}^{M} z_n\right|^2\right)\sum_{n'=1}^{M}\left(\frac{z_{n'}}{z}+\frac{\bar{z}_{n'}}{\bar{z}}\right)\right]|z|^{-4}$$

$$+ \mathcal{O}\left(|z|^{-6}\right) \ ,$$

Effective size

$$|c_{\text{effective}}|^2 = \frac{1}{2M}\sum_{n=1}^{M}\left(|z_n|^2 + 2|c_n|^2\right) - \frac{1}{2M^2}\left|\sum_{n=1}^{M} z_n\right|^2 ,$$

Figure – fractional Chern-Simons vortex



Future developments

- monopole-vortex systems
- the GNOW-duality
- group theory of vortices (in preparation)
- Yang-Mills-Chern-Simons-Higgs (in preparation)
- non-BPS corrections / stability
- Q-lumps in SO, USp theories
- quantum corrections to the $\mathcal{N} = 1$ Kähler quotients
- Ricci-flat Calabi-Yau metrics
- D-brane constructions
- knotted solitons
- quantized vortices
- domain wall systems in SO, USp theories
- the mass deformed theories
- etc.

Thanks for your attention