(A few) Virtues and Mysteries of SUSY $\mathcal{N} = 4$

A guided tour along Wilson-Maldacena Loops (WML) and their low-dimensional kin

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Supersymmetric WML : CONJECTURES

1. Bern-Korchemsky duality (dual conformal symmetry): WML $\leftrightarrow$ MHV(maximally helicity violating) gluon amplitudes
2. Maldacena duality: WML $\leftrightarrow$ strings in AdS_5 X S^5
3. SUSY $\mathcal{N} = 4$—YM_2 duality: WML on S^2 $\leftrightarrow$ Matrix Models $\leftrightarrow$ YM_2

I will concentrate on item 3, following

- A.B., L.Griguolo, F.Pucci and D.Seminara, JHEP 0806:083 (2008)[BGPS]
- A.B., L.Griguolo, F.Pucci, D.Seminara, S.Thambyahpillai and D.Young, JHEP 0908:061 (2009)[BGPSTY1]
- A.B., L.Griguolo, F.Pucci, D.Seminara, S.Thambyahpillai and D.Young, JHEP 0908:061 (2009) [BGPSTY2]

Plan of the talk

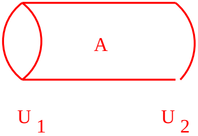
- Digression on YM_2: WL and WL correlators on a sphere. Expansions in terms of characters and of instantons.
- The zero-instanton sector and its relation to a perturbative treatment.Generalizing to loop correlators $\rightarrow$ multimatrix models.
- Supersymmetric WM loops on S^2: the DGRT coupling.
- The relation between YM_2 and SUSY $\mathcal{N} = 4$: weak coupling tests on loops and loop correlators.
- A comment on the strong coupling situation.

The Wilson Loop in YM_2

- YM_2: an almost topological theory (invariance under area-preserving diffeomorphisms) [Witten]
- Basic ingredient: the cylinder amplitude (heat kernel propagator)
- Expansion in terms of characters χ:

$$\mathcal{K}(A; U_1, U_2) = \sum_R \chi_R(U_1)\chi_R^\dagger(U_2)e^{-\frac{g^2 A}{2}C_2(R)}.$$

A$=$ area of the cylinder, sum over the representations R of $U(N)$, $C_2(R)$ quadratic Casimir.



Poisson transformation

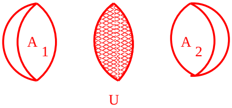
Poisson Transformation → dual expansion in terms of instantons [Gross-Matytsin]

$$\mathcal{K}(A; U_1, U_2) = \sum_P \frac{(g^2 A)^{-N/2}}{J(\theta_i)J(\phi_i)} \sum_l (-1)^{P+(N-1)\sum l_i}$$

$$exp\left(-\frac{1}{2g^2A}\sum_{i=1}^{N}(\phi_i - \theta_{P(i)} + 2\pi i l_i)^2\right).$$

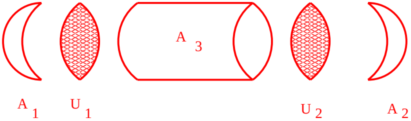
$\{e^{i\theta_i}\}, \{e^{i\phi_i}\}$ eigenvalues of U_1 and U_2 and $J(\theta_i) = \Pi_{i<j} 2sin(\frac{\theta_i - \theta_j}{2})$.

Single WL



$$\int dUK(A_1;\mathbf{1},U)\,TrUK(A_2;U,\mathbf{1})$$

WL Correlator



$$\int dU_1 dU_2 K(A_1;\mathbf{1},U_1)\,TrU_1 K(A_3;U_1,U_2)\,TrU_2 K(A_2;U_2,\mathbf{1})$$

$$\mathcal{W}(A_1, A_2) = \frac{(g^4 A_1 A_2)^{-\frac{N^2}{2}} (g^2 A_3)^{-\frac{N}{2}}}{\mathcal{Z} N^2} \sum_P \sum_s (-1)^{P+(N-1)\sum s_i}$$

$$\int_{R^{2N}} d^N\theta d^N\phi \left(\sum_{r,s=1}^{N} e^{i\theta_r + i\phi_s} \right) \Delta(\theta_i)\Delta(\phi_i) \times$$

$$\exp\left(-\frac{1}{2g^2 A_1} \sum_{i=1}^{N} \theta_i^2 - \frac{1}{2g^2 A_3} \sum_{i=1}^{N} (\phi_i - \theta_{P(i)} + 2\pi i s_i)^2 - \frac{1}{2g^2 A_2} \sum_{i=1}^{N} \phi_i^2 \right)$$

The zero instanton case: $s_i = 0, \ \forall i \rightarrow$ multimatrix model.

$$\mathcal{W}^{(0)}(A_1, A_2) = \frac{1}{C_N N^2} \int DV_1 DV_2 \mathrm{Tr}(e^{i\sqrt{g^2 A_1} V_1}) \mathrm{Tr}(e^{i\sqrt{g^2 A_2} V_2})$$
$$\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(V_1^2) - \frac{1}{2}\mathrm{Tr}(V_2^2) - \frac{1}{2A_3}\mathrm{Tr}((\sqrt{A_1} V_1 - \sqrt{A_2} V_2)^2)},$$

Total area of the sphere $A = A_1 + A_2 + A_3$,V_1, V_2 hermitean matrices, $C_N \equiv \mathcal{W}^{(0)}(g = 0)$. Generalization to many loop correlators straightforward.

The connected correlator $\mathcal{W}^{(0)}(A_1, A_2) - \mathcal{W}^{(0)}(A_1)\mathcal{W}^{(0)}(A_2)$ explicitly computed in terms of Laguerre polynomials. Analytic function of the coupling g and of rescaled areas. Decompaction limit of the sphere $A \to \infty$: $\mathcal{W}^{(0)}$ perturbative expansion = results of Feynman graphs. Small coupling check of $YM_2 \leftrightarrow$ SYM after replacement $g^2_{2dim} = -g^2_{4dim}/A$.

Strong Coupling in YM$_2$

WL and WL correlators in the zero instanton sector → matrix model (Laguerre polynomials). At large N

$$\mathcal{W}^{(0)}(A) \simeq \frac{1}{\sqrt{g^2 NA}} \mathrm{J}_1(\sqrt{g^2 NA})$$

J_1 Bessel function. Setting $g^2 NA \equiv -\lambda >> 1$ (for future purposes!)

$$\mathcal{W}^{(0)}(\lambda) \simeq \frac{1}{\sqrt{\lambda}} \mathrm{I}_1(\sqrt{\lambda}) \simeq \frac{\mathrm{e}^{\sqrt{\lambda}}}{\lambda^{\frac{3}{4}}}$$

and

$$\frac{\mathcal{W}^{(0),conn.}_{1,2}}{\mathcal{W}^{(0)}_1 \mathcal{W}^{(0)}_2} \simeq \frac{\lambda \tilde{A}_1 \tilde{A}_2}{N^2 A^2} \sum_{k=1}^{\infty} k \left(\sqrt{\frac{A_1 A_2}{\tilde{A}_1 \tilde{A}_2}} \right)^{k+1}$$

where $A = A_1 + A_2 + A_3$ and $\tilde{A}_{1,2} = A - A_{1,2}$.

Wilson-Maldacena Loop in SUSY N=4

$$\mathcal{W}_{D=10}(C) = \frac{1}{N} Tr \left[P \exp(i \oint_C \mathcal{A}_a(X) dX^a) \right],$$

$a = 0, ..., 9.$
Dimensional reduction $\rightarrow$

$$\mathcal{W}_{D=4}(C) = \frac{1}{N} Tr \left[P \exp(i \oint_C ds (\mathcal{A}_\mu(x) \dot{x}^\mu(s) + \Phi_I(x) \theta^I |\dot{x}|) \right],$$

θ^I unit six-vector $\theta^I \theta^I = 1,\ I = 4, ..., 9$ in order to preserve **local** supersymmetry.
Fermion couplings may be explored; not considered here.
In Euclidean formulation $|\dot{x}| \rightarrow i|\dot{x}|$. The loop no longer a phase factor.

To enforce **global** supersymmetry (number of preserved charges)

- Choose particular contours
- Tune the scalar couplings $\rightarrow \theta^I(s)$

For $\theta^I = const.$ C infinite straight line (1/2 BPS) $\rightarrow \mathcal{W} = 1!$

Focus our attention on a circular loop of radius R

$$x_\mu = (R\, cos\, t_1\; R\, sin\, t_1, 0, 0).$$

With $\theta^I \theta^I = 1$ the effective propagator $(\mathcal{A}, \Phi)$ becomes

$$\begin{aligned} &\langle (i\mathcal{A}^a + \theta\Phi^a)_{t_1} (i\mathcal{A}^b + \theta\Phi^b)_{t_2} \rangle \\ &= \frac{g^2 \delta^{ab}}{4\pi^2} \frac{|\dot{x}_1||\dot{x}_2| - (\dot{x}_1 \cdot \dot{x}_2)}{(x_1 - x_2)^2} = \frac{g^2\delta^{ab}}{8\pi^2} = const. \end{aligned}$$

Perturbative diagrams **without interactions** $\rightarrow$ matrix model

$$\mathcal{W}_{circle} = \frac{1}{N} L^{(1)}_{N-1}(-g^2/4)\exp(g^2/8),$$

Laguerre polynomial $L^{(1)}_{N-1}$.

Interactions: cancel at $\mathcal{O}(g^4)$!
Should it persist at higher orders, matrix model exact result!
Conjecture: SYM $\leftrightarrow$ YM_2. At large N and large λ

$$\mathcal{W}_{circle} \simeq e^{\sqrt{\lambda}}$$

Agreement with AdS/CFT (saddle point).
However for non-smooth contours $\rightarrow$ cusp singulatities.

Topological Twist

Zarembo's proposal:

$$x^\mu \to x^\mu M^I_\mu, \qquad \theta^I = M^I_\mu \frac{\dot{x}^\mu}{|\dot{x}|}, \qquad M^I_\mu M^I_\nu = \delta_{\mu\nu}.$$

Euclidean WML

$$\mathcal{W}(C) = \frac{1}{N} Tr\, P \oint_C dx^\mu (iA_\mu + M^I_\mu \Phi_I)$$

The *effective* propagator vanishes

$$\langle (iA_\mu + M^I_\mu \Phi_I)(iA_\nu + M^K_\nu \Phi_K) \rangle = (i^2 \delta_{\mu\nu} + M^I_\mu M^I_\nu) D(x-y) = 0!$$

Contributions from diagrams without internal vertices vanish. Up to $\mathcal{O}(g^4)$ interacting diagrams vanish as well (Zarembo). From loop equations proof at any order (Guralnik & Kulik) $\to$ $\langle \mathcal{W}(C) \rangle = 1$. Too simple!

DGRT proposal

Drukker, Giombi, Ricci and Trancanelli (DGRT) proposed the choice

$$M^I_\mu = -\sigma^j_{\mu\nu}\mathcal{M}^j_I x_\nu, \qquad j = 1,2,3.$$

The quantities $\sigma^j_{\mu\nu}$ are 't Hooft symbols, the matrix $\mathcal{M}^j_I$ is 3x6, with $\mathcal{M}^k_k = 1$, $k = 1,2,3$ and zero otherwise.

$$\mathcal{W}(C) = \frac{1}{N} Tr\, P \oint_C dx^\mu (iA_\mu + M^k_\mu \Phi_k)$$

Only three scalars coupled. Need of a length-scale. The loop is supersymmetric only when restricted on a three-dimensional sphere.

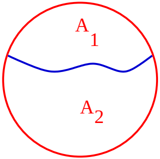
Restriction on S^2

The effective propagator in 2ω dimensions

$$\Delta_\omega = \frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{(\dot{x}_1\cdot\dot{x}_2)(x_1\cdot x_2-1)-(x_1\cdot\dot{x}_2)(\dot{x}_1\cdot x_2)}{(x_1-x_2)^{2(\omega-1)}}$$

No special features; **less singular** at coincident points (no "cusp anomaly"); the singularity is integrable on the contour. For a smooth generic contour (DGRT) (1/8 BPS)

$$\mathcal{W}(A_1,A_2) = 1+\lambda\frac{A_1\,A_2}{A^2}+\mathcal{O}(\lambda^2)$$



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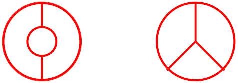
CONJECTURE

Is the theory equivalent to a matrix model?

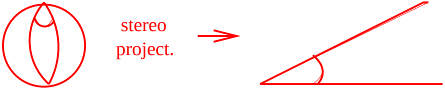
$$\mathcal{W}(A_1, A_2) = \frac{1}{\mathcal{Z}\,N} \int dM\, Tr(e^M) exp\left(-\frac{NA^2}{\lambda A_1 A_2} Tr(M^2) \right)$$

Perturbative check at $\mathcal{O}(\lambda^2)$ (BGPS).



Diagrams with interactions (bubble + spider): each one UV divergent (no matter the contour) but their sum is not.

Test for a particular cusped contour: two longitudes on the sphere (1/4 BPS)



Predictions:

- At $\mathcal{O}(\lambda)$: $\frac{\lambda}{8\pi^2}\delta(2\pi-\delta) \propto \frac{A_1 A_2}{A^2}$
  Result OK, but somehow trivial (no interaction).

- At $\mathcal{O}(\lambda^2)$: $-\frac{\lambda^2(N^2-1)}{384N^2\pi^4}\delta^2(2\pi-\delta)^2$

δ=angle between longitudes.

At this order the test is much more difficult.

UV singularities (in Feynman gauge) exactly cancel between self-energy and triple-vertex diagrams.

A few final integrals only **numerically**.

The plot fits the polynomial of the prediction astonishingly well (error $< 10^{-8}$ in the entire range of δ). [BGPS]

No cusp singularity.
Contribution from interactions **essential**.

What about WML correlators?

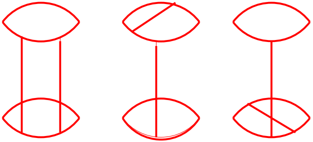
Remember predictions in YM_2:
Connected normalized WML correlators on the sphere $\rightarrow$ analytic functions of the rescaled areas. Gaussian (multi)-matrix model.
Can we extend the conjecture $YM_2 \leftrightarrow$ SUSY $\mathcal{N}$=4 to WML correlators?
Consider coaxial loops on the sphere at different latitudes (1/8 BPS).

$$\mathcal{O}(\lambda) = \mathcal{W}^{(1)}_{1,2} = \frac{\lambda}{N^2}\frac{A_1 A_2}{A^2}$$

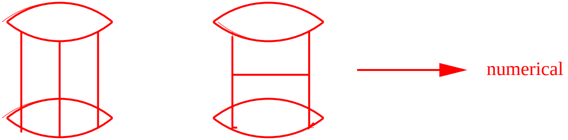
OK with conjecture.

$$\mathcal{O}(\lambda^2) = \mathcal{W}^{(2)}_{1,2} = \frac{\lambda^2}{2N^2A^4}A_1A_2(A_1A_3 + A_2A_3 + 3A_1A_2),$$

in agreement, but somehow trivial (no interaction; see diagrams)



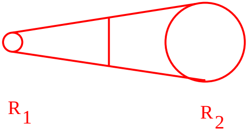
At $\mathcal{O}(\lambda^3)$ several diagrams contribute, ladder and diagrams with interactions. Examples:



ladder interacting

Complete analytical evaluation impossible. Again a few integrals numerically. Agreement with the conjecture with error $< 10^{-4}$ in two configurations (symmetric and asymmetric) for a sizeable range of latitudes (BGPSTY2).

A different criterion: to shrink one of the two circles (Arutyunov, Plefka, Staudacher, JHEP 0112:014, 2001, [APS]) $R_1 \to 0$



OPE for the single loop

$$\frac{\mathcal{W}_1}{\langle \mathcal{W}_1 \rangle} = 1 + \sum_k C_k R_1^{\Delta_k} O^{(k)}$$

where $O^{(k)}$ basis of local gauge invariant operators, Δ_k their dimensions, C_k coefficients. Perturbative expansion $\Delta_k = \Delta_k^{(0)} + \lambda \Delta_k^{(1)} + \dots$ Introducing OPE in a normalized connected correlator

$$\frac{\langle \mathcal{W}_1 \mathcal{W}_2 \rangle}{\langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle} = 1 + \sum_k C_k R_1^{\Delta_k} \frac{\langle O^{(k)} \mathcal{W}_2 \rangle}{\langle \mathcal{W}_2 \rangle}$$

Hunting for logarithms

Expanding

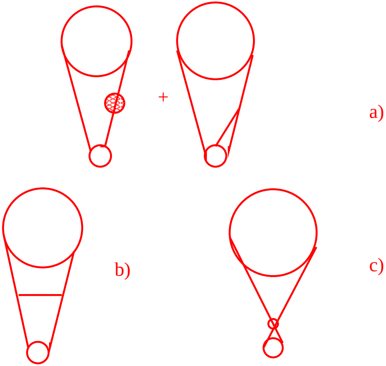
$$R_1^{\Delta_k} = R_1^{\Delta_k^{(0)}} (1 + \Delta_k^{(1)} log\ R_1 + \ldots$$

several logarithmic contributions are produced. They are **incompatible** with a matrix model; jeopardize YM_2– SUSY correspondence!

In [APS] normal WML ($\theta^i = const$). Contributions $\propto \lambda^3 R_1^2 log\ R_1 + \ldots$ YM_2–SUSY correspondence is OUT.

Test with [DGRT] coupling: On S^2 effective gauge connection $\mathcal{A}_i^{eff} = \mathcal{A}_i + i\epsilon_{ijk} x^j \frac{\Phi^k}{R}$
Diagrams with no interaction cannot produce logarithms.

At $\mathcal{O}(\lambda^3)$ consider the following diagrams



- Diagrams a)

$$-\frac{R_1^4}{256\pi^2} logR_1 cos^4\frac{\theta_0}{2},$$

- Diagram b)

$$-\frac{R_1^4}{384\pi^2} logR_1 cos^4\frac{\theta_0}{2},$$

- Diagram c)

$$\frac{5R_1^4}{768\pi^2} logR_1 cos^4\frac{\theta_0}{2}.$$

On the sphere $R=1$, $R_1 \propto sin\theta_0$, $R_2 \propto cos\frac{\theta_0}{2}$.
The sum vanishes! Conjecture OK at $\mathcal{O}(\lambda^3)$! [BGPSTY1]

This result points towards the existence of a "protected" operator:

$$\mathcal{O}(x_0) = \mathrm{Tr}[2R\Phi_3 - iR^2 F_{12} - R^2(\partial_1\Phi_1 + \partial_2\Phi_2)]^2,$$

x_0 being the shrinkage point (the center of the small circle).

With the [DGRT] coupling to the contour of the effective connection $\mathcal{A}^{eff}$, operators of different classical dimensions couple with the same power of the parameter which set the size of the shrinking latitude, $R_1 \propto sin\theta_0$ in this case
The radius R of the sphere cancels in the results (conformal invariance).

Thereby a cancellation can occur among terms which behave differently in the usual coupling situation [APS].

This is indeed what happens here!

A comment on the Strong Coupling

In the strong coupling limit, the YM_2 prediction to be compared with SUGRA in the large separation limit of the loops

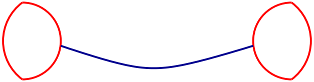
$$\frac{\mathcal{W}_{1,2}^{conn.}}{\mathcal{W}_1\mathcal{W}_2} \simeq \frac{2\lambda\tilde{A}_1\tilde{A}_2}{N^2A^2}\left(\frac{A_1A_2}{\tilde{A}_1\tilde{A}_2}\right)^{3/2}$$

where $A = A_1 + A_2 + A_3$ and $\tilde{A}_{1,2} = A - A_{1,2}$. In such configuration

$$\frac{\mathcal{W}_{1,2}^{conn.}}{\mathcal{W}_1\mathcal{W}_2} \simeq \frac{\lambda}{8N^2}\frac{\theta_0^3\theta_1^3}{4},$$

the (small) angles $\theta_{0,1}$ being related to the loop sizes.

Several possible contributions have been explored in SUGRA [BGPSTY2] (after using the AdS/CFT conjecture). Only a qualitative agreement (coefficients do not match).



Lightest gravitational modes (large distances).

Possible errors in previous normalizations in the literature? Different combinations of contributing operators? (remember that the [DGRT] coupling was essential in establishing the YM-SUSY correspondence).
An open problem.

For curious people:

Fluctuations of R-R 5-form and of the metric

$$\frac{\lambda}{8N^2}\frac{\theta_0^3\theta_1^3}{4}x3$$

Fluctuations of NS-NS B-field (legs in S^5 directions)

$$-\frac{\lambda}{8N^2}\left(\frac{\theta_0^3\theta_1^3}{8}-\frac{\theta_0^3\theta_1^4+\theta_0^4\theta_1^3}{5}\right)1/2$$

Fluctuations of NS-NS B-field (legs in AdS_5 directions)

$$-\frac{\lambda}{8N^2}\left(\frac{3\theta_0^3\theta_1^3}{8}+\frac{\theta_0^3\theta_1^4+\theta_0^4\theta_1^3}{5}\right)\sqrt{2}$$

Yang-Mills

$$\frac{\lambda}{8N^2}\frac{\theta_0^3\theta_1^3}{4}$$