Quantum Gravity and Noncommutative Spacetimes

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A. P. Balachandran

Physics Department
Syracuse University
Syracuse, NY 13244-1130
USA

Work in progress with Alberto Ibort, Giuseppe Marmo and Mario Martone.

It has been argued by Doplicher, Fredenhagen and Roberts and other authors that the algebra of functions on spacetime is noncommutative at Planck scales.

The basic idea is that attempts to probe Planck scale physics will create black holes.

The horizons of these black holes will then prevent us from probing such length and time scales.

Now quantum gravity has excitations called quantum geons. [Friedman - Sorkin].

They have extraordinary statistical properties.

Also, without topology change, a geon has no anti-geon.

That is, no anti-particle.

We argue that spacetime noncommutativity can emerge from geons.

Geons are constructed from prime manifolds.

They are used as fixed time spatial slices for geons.

What are prime manifolds?

If M_1 and M_2 are two manifolds in d dimensions, their connected sum $M_1 \# M_2$ is defined as follows:

Remove d-dimensional balls B_1, B_2 from M_1, M_2.

Then

Boundary of $M_i/B_i = \text{sphere } S_i^{d-1}$.

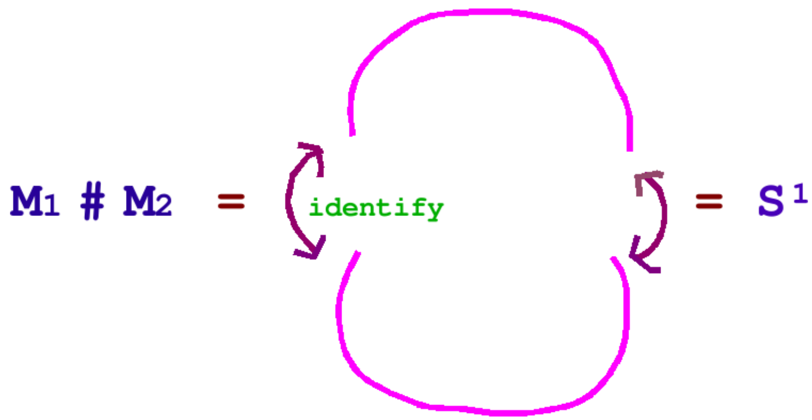
Attach M_1/B_1 to M_2/B_2 by identifying the spheres S_i^{d-1}.

The result is

$$M_1 \# M_2.$$

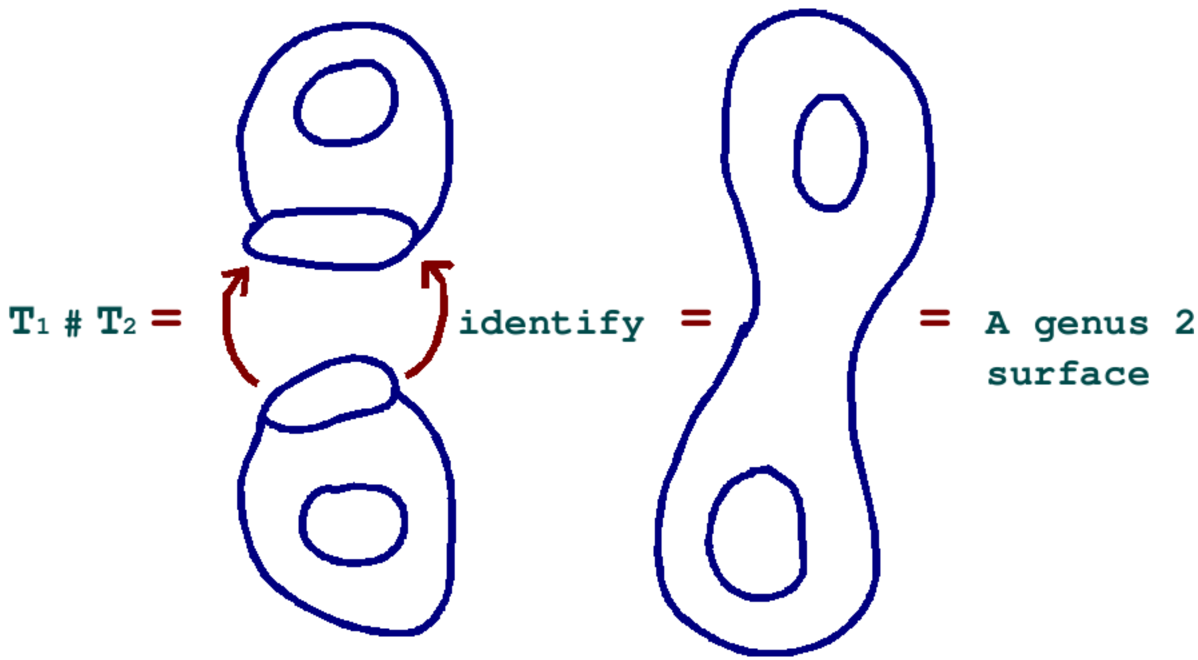
Example 1:

$d = 1$, B_i = intervals, $M_1 = S^1$, $M_2 = S^1$



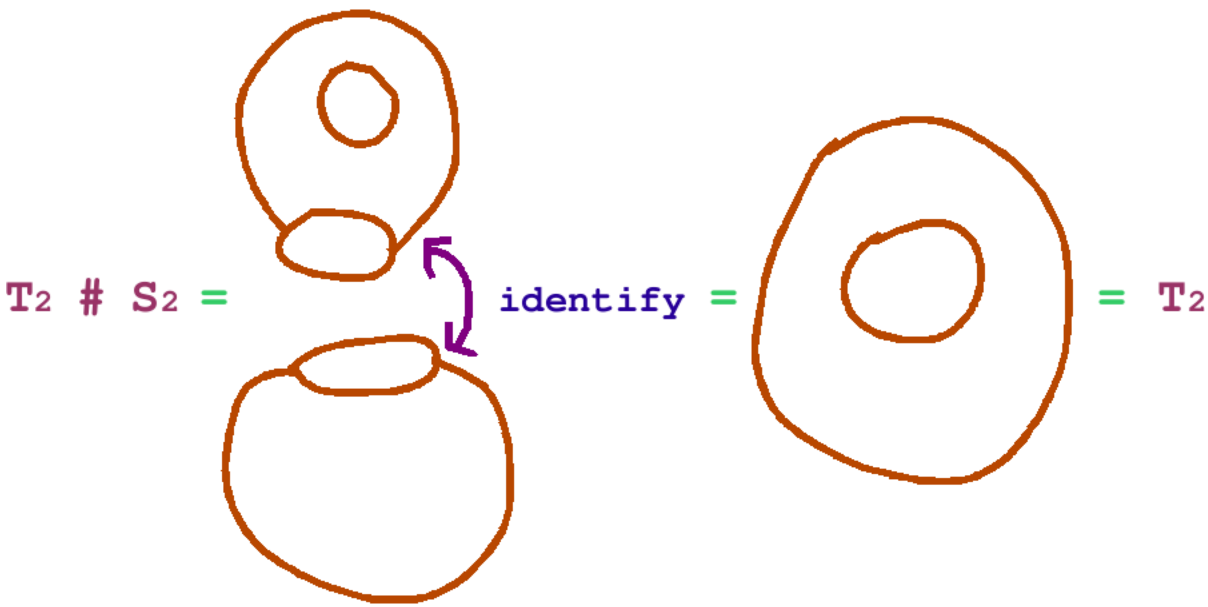
Example 2:

$d = 2$, $M_1 = T^2$, $M_2 = T^2$, B_i = discs.



Example 3:

$M_1 = T^2$, $M_2 = S^2$



In general

$$M \# S^d = M.$$

Prime Manifolds: P_α

They are building blocks of all manifolds in 2 and 3 dimensions.

Let M = A manifold with one asymptotic region.

Then, $M = R^d \# P_1 \# P_2 \# \cdots \# P_k$,

$P_\alpha \neq S^2$, $d = 2, 3$.

The decomposition is unique.

For $d = 2$

Primes: T^2 (orientable), RP^2 (non-orientable).

For $d = 3$

Infinite in number, not classified.

Example 1: Spherical spaces

Let $S^3 \approx SU(2)$.

Let H, H', discrete subgroups of $SU(2)$ such that the action

$$g \longrightarrow hgh'^{-1},\ h \in H,\ h' \in H'$$

of $\mathcal{H} = (H \times H')/Z_2$ is free. (No Z_2 needed if H or H' does not contain Z_2.)

Then

$S^3/\mathcal{H}$ = spherical spaces = primes

Examples are Lens spaces $L_{p;q}$:

$H = Z_p$, $H' = Z_q$, p and q relatively prime.

Example 2: Hyperbolic spaces

$$H^3 = \langle x \in R^4 : (x^0)^2 - (\vec{x})^2 = 1, x^0 > 0 \rangle = \text{upper sheet of mass hyperboloid.}$$

Lorentz group $L_+^\uparrow$ acts on H^3.

Let D any discrete subgroup of $L_+^\uparrow$ acting freely on H^3.

$$H^3/D = \text{prime.}$$

Example 3:

$T^2 \times T^1$, $T^2 \hat{\times} T^1$ = Twisted T^1 bundle on T^2 etc.

Quantum gravity on these manifolds give quantum geons.

Symmetry Groups of Geons

Let $\mathcal{D}$ be the diffeomorphism group of a spatial slice M.

It has several subgroups.

$\mathcal{D}^{\infty} \subset \mathcal{D}$ = $\{$ asymptotically trivial diffeos $\}$.

$\mathcal{D}^{\infty}_{0}$ = $\{$ connected component of $\mathcal{D}^{\infty}$ $\}$.

The Gauss law constraint in quantum gravity $\Rightarrow$ $\mathcal{D}^{\infty}_{0}$ acts trivially on quantum states.

So

“Symmetry group” in quantum gravity = $\mathcal{D}/\mathcal{D}_0^\infty$.

We focus on

“Mapping Class Group” $\equiv \mathcal{D}^\infty/\mathcal{D}_0^\infty \subset \mathcal{D}/\mathcal{D}_0^\infty$

For one-geon manifolds, generally it reflects the internal diffeos of the primes:

$$\left(\mathcal{D}^\infty/\mathcal{D}_0^\infty\right)\left(R^d \# P_\alpha\right) \approx Mapping\ class\ group\ of\ P_\alpha.$$

They will influence the twists in the spacetime algebra.

Examples

For appropriate spherical spaces,

$$\begin{aligned} \mathcal{D}^{\infty}/\mathcal{D}_0^{\infty} &= Z_n \\ &= D_{2n} = \text{Dihedral group} \\ &= T = \text{Tetrahedral group} \\ &= O = \text{Octahedral group} \\ &= I = \text{Icosahedral group} \end{aligned}$$

and their products with Z_2.

They come from the fact that

Spherical space = $H \backslash SU(2)/H' = \langle HgH' | g \in SU(2) \rangle$.

So if $\mathcal{N}_H$, $\mathcal{N}_{H'}$ normalizers of H, H' in $SU(2)$,

$$\mathcal{N}_H = \langle g \in SU(2) : gHg^{-1} = H \rangle$$

$$\mathcal{N}_{H'} = \langle g \in SU(2), \;\; gH'g^{-1} = H' \rangle$$

$$\mathcal{D}^\infty / \mathcal{D}_0^\infty \approx \left[\left(\mathcal{N}_H / H \right) \times \left(\mathcal{N}_{H'} / H' \right) \right]$$

If

$A \equiv \times_i Z_{n_i} \subset \mathcal{D}^\infty / \mathcal{D}_0^\infty$

is maximal abelian subgroup of $\mathcal{D}^\infty / \mathcal{D}_0^\infty$, we will “Drinfel’d twist on A”.

STATISTICS

Consider a state of two identical geons localized at points P_1 and P_2.

Its quantum state is

$$\Psi_{P_1} \otimes \Psi_{P_2}$$

The simple flip operator τ_0

$$\tau_0 \left(\Psi_{P_1} \otimes \Psi_{P_2} \right) = \Psi_{P_2} \otimes \Psi_{P_1}$$

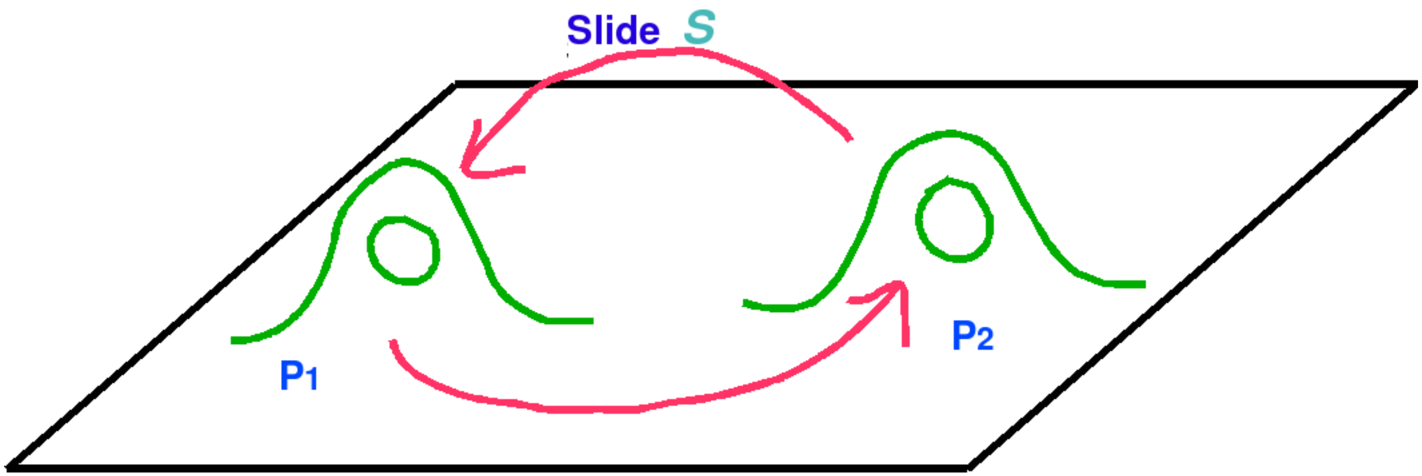
fulfills

$$\tau_0^2 = id$$

It would be the naive choice for exchange, but is not the general one possible.

We can follow up τ_0 with an internal diffeo d_i ($i = 1, 2$) of either geon.

Or we can define an operator called slide S. It is a diffeo corresponding to moving geon 2 along a noncontractable loop threading geon 1.



Then

$$\tau_0 S$$

is also an exchange.

Statistics in two geon sector $R^d \# P \# P$ is thus governed by the group

$$\left(\mathcal{D}^\infty / \mathcal{D}_0^\infty\right)(R^d \# P \# P).$$

This group can be quite non-abelian.

Not clear how to quantize all of it, go to N-geon sector.

So we will twist τ_0 using A.

It will incorporate abelian internal diffeos.

It will lead to noncommutative spacetimes at Planck scales.

Drinfel'd Twist for A.

For Poincaré group, the Drinfel'd twist is

$$F_\theta = e^{\frac{i}{2}\, P_\mu \otimes \theta^{\mu\nu} P_\nu}$$

It twists flip τ_0 to

$$\tau_\theta = F_\theta^{-1} \tau F_\theta = F_\theta^{-2} \tau$$

On plane waves e_p, $e_p(x) = e^{ipx}$,

$$F_\theta e_p \otimes e_q = e^{-\frac{i}{2}\, p_\mu \theta^{\mu\nu} q_\nu} e_p \otimes e_q$$

This is F_θ on irreducible representations (IRR's) of translations. We generalize this to the abelian A.

Consider first Z_n

$$Z_n = \{e, z = e^{i2\pi/n}, \cdots e^{i(n-1)2\pi/n}\}.$$

It has IRR's

$z \to z^m$, $m = 0, 1, \cdots (n-1)$

with characters χ_m

$$\chi_m(z) = e^{i(2\pi/n)m}.$$

Then on space where Z_n acts,

$$P_m = \frac{1}{n}\sum_{k=0}^{n-1}\overline{\chi}_m(z^k)z^k$$

is projector to IRR $z \longrightarrow \chi_m(z)$:

$$\begin{aligned} z^l P_m &= \frac{1}{n}\sum_0^n \overline{\chi}_m(z^k)z^{k+l} \\ &= \frac{1}{n}\sum_0^n \overline{\chi}_m(z^{k-l})z^k \\ &= \chi_m(z^l)P_m \end{aligned}$$

since

$$\overline{\chi}_m(z^{k-l}) = \chi_m(z^l)\overline{\chi}_m(z^k).$$

If

$$A = Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k},$$

its IRR's are

$$\vec{m} = (m_1, m_2, \cdots m_k),\ 0 \leq m_l \leq n_l - 1$$

with characters

$$\chi_{\vec{m}} = \prod_i \chi_{m_i}$$

and projectors

$$P_{\vec{m}} = \prod P_{m_i},\ \sum_{\vec{m}} P_{\vec{m}} = \mathbf{1}.$$

The Drinfel'd twist F_θ for A is

$$F_\theta = \sum_{\vec{m},\vec{m}'} e^{\frac{i}{2} m_i \theta_{ij} m'_j} P_m \otimes P_{m'} \tag{1}$$

Remark:

IRR's do not change if $m_l \to m_l + n_l$.

So

$$\theta_{ij} = \frac{4\pi}{n_{ij}}, \frac{n_i}{n_{ij}}, \frac{n_j}{n_{ij}} \in Z.$$

So θ_{ij} is quantized if A is finite.

This is unlike the case for Moyal plane.

$\Big\{$But twisting $S^1 \times R$ quantizes θ.

(Bal, Govindarajan, Martins, Teotonio-Sobrinho)$\Big\}$

TOWARDS QUANTUM GRAVITY

Steps:

First specify

1) Spacetime algebra, its Hopf symmetries.

2) Its symmetrization/anti-symmetrization compatible with Hopf symmetries.

Then formulate

3) Its differential geometry, gravity (see Aschieri et al.)

Finally

4) Quantize.

We are at step 1).

SPACETIME ALGEBRA

We first find a model for τ_θ, the twisted flip operator.

Consider

$$\tau_\theta = F_\theta^{-1} \tau_0 F_\theta$$

It incorporates abelian internal diffeos.

So it reflects spacetime topology.

Coproduct on diffeos d must commute with τ_θ:

$$\Delta_\theta(d)\tau_\theta = \tau_\theta \Delta_\theta(d).$$

Coproduct on diffeos is therefore changed:

$$\Delta_\theta(d) = F_\theta^{-1}\Delta_0(d)F_\theta,$$
$$\Delta_\theta(d) = d \otimes d, \quad d \in \mathcal{D}^\infty/\mathcal{D}_0^\infty.$$

For Hopf action of diffeos on spacetime algebra, we then require on multiplication map:

$$m_\theta(f \otimes g) = m_0\left(F_\theta^{-1} f \otimes g\right)$$
$$m_0(\alpha \otimes \beta) = \alpha\beta = \text{Pointwise multiplication map.}$$

So spacetime is noncommutative.

REMARKS

– As diffeos are asymptotically flat, noncommutativity is at Planck scales (size of geons).

– This noncommutative algebra is associative (as Δ_θ is coassociative).

CONJECTURE

1) Incorporation of non-abelian diffeos in $F_\theta \Rightarrow$ Quasi-Hopf algebra $\Rightarrow$ Nonassociative noncommutative spacetimes.

Geons: Precursors of Black Holes?

Gannon, Friedman, Schleich, Witt:

Geon manifolds classically will evolve into singular manifolds (black holes?).

$\Rightarrow$ Not clear if their topology will survive quantization.

Thus if quantum gravity with self-adjoint Hamiltonian exists, time evolution for all time too will exist.

But through a glass darkly:

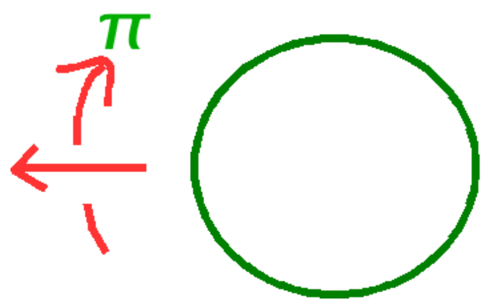
We can see that there is a connection between spacetime noncommutativity and black holes (Doplicher, Fredenhagen, Roberts).

KNOTS IN 3d

These can model above features of geons. (Bal, Ajit Srivastava, Sumati Surya)

Example:

1 Unknot

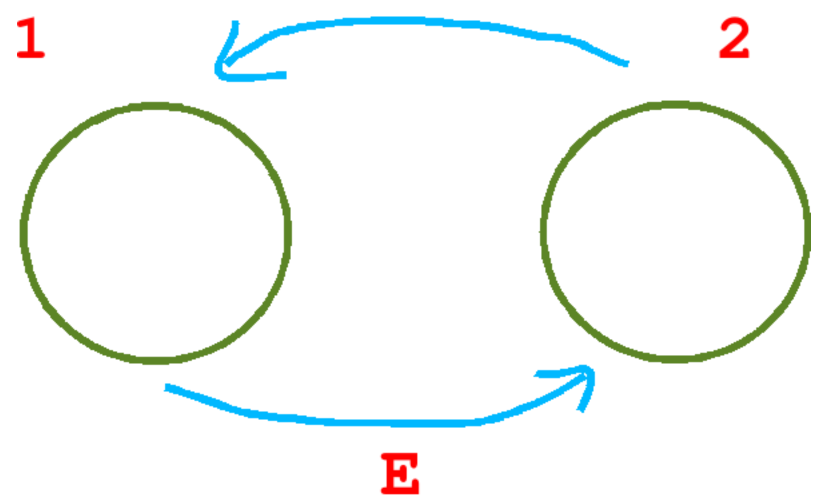


Take it to be on a plane. Rotate by π as shown. We get back unknot.

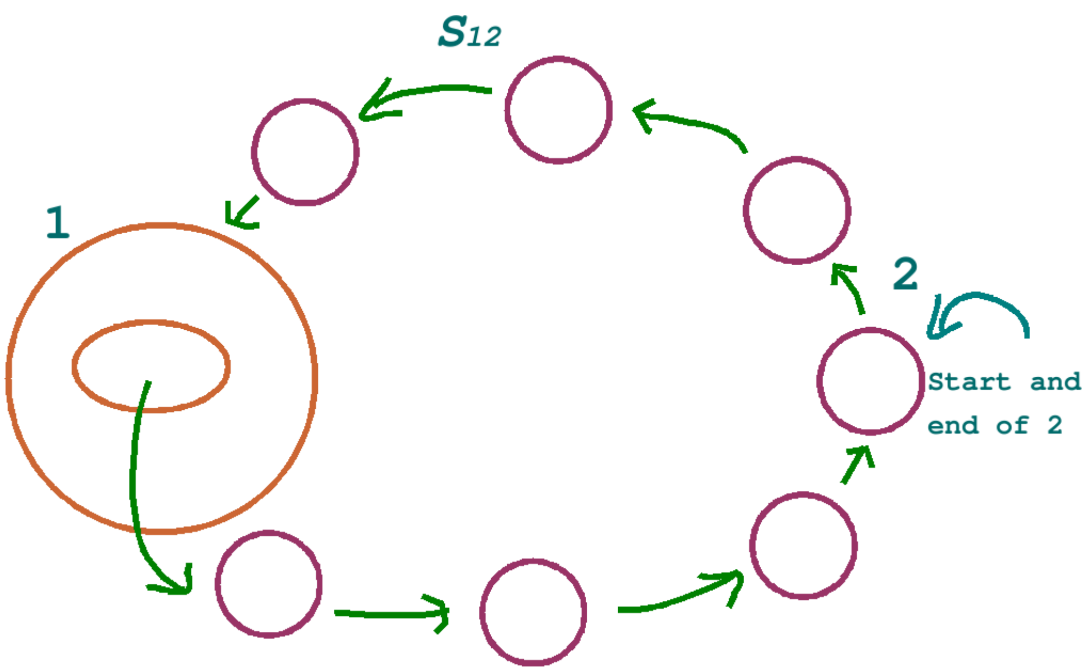
Gives a nontrivial diffeo R_π.

$$\left[\ (R_\pi)^4 = R_{4\pi} \in \mathcal{D}_0^\infty .\ \right]$$

2 Unknots



E = simple exchange



S12: Slide of 2 through 1

So statistics group involves E, S_{12}, $R_\pi^{(1)}$, $R_\pi^{(2)}$ (internal diffeos of knots 1, 2).

The situation is rich for more complex knots ("Motion Groups" of D. Goldsmith).

REMARK ON SPACETIME NONCOMMUTATIVITY

In above, time and space commute.

Not clear how to get their noncommutativity naturally.

QUANTIZATION

Let us assume that we are given a quantum gravity on commutative spacetimes ($\theta_{ij} = 0$).

Let ϕ_0 be a quantum field for $\theta_{ij} = 0$.

Then general considerations (see our papers) show that the twisted quantum field is obtained by a dressing transformation (Grosse, Faddeev-Zamolodchikov):

$$\phi_\theta = \phi_0 \left[\sum_{\vec{m},\vec{m}'} e^{-i/2 m_i \theta_{ij} m'_j} \overleftarrow{P}_{\vec{m}} \otimes \overrightarrow{P}_{\vec{m}'} \right]$$

where $P_{\vec{m}}$ acts to left, $P_{\vec{m}'}$ to right.

If ϕ_θ has eigenvalues $\vec{m}$ under $A = \times_i Z_{n_i}$,

so that $\phi_\theta = \phi_\theta^{\vec{m}}$,

the above becomes

$$\phi_\theta^m = \sum_{\vec{m}'} \phi_0^{\vec{m}} e^{-\frac{i}{2} m_i \theta_{ij} m'_j} P_{\vec{m}'}$$

ON CAUSALITY

The diffeos in A are asymptotically flat.

Thus they are nontrivial only in a volume of the size of the geon.

That means that spacetime noncommutativity is confined to a Planck volume.

Spacetime noncommutativity induces acausal effects [Bal et al.]

In this gravity model, these are confined to a Planck volume.

They are not macroscopically present.

FINAL REMARKS

- We naturally get only space-space noncommutativity.
  It will be good to naturally obtain also time-space noncommutativity.

- The twists are quantized since the twisted group A is finite.
  There are examples where A contains factors of $Z : A = Z \times Z \times \cdots$.
  Then θ_{ij} is not quantized and we get analogues of Moyal plane.

- Noncommutativity and acausality are confined to Planck scales.