Superconductivity and the Riemann zeros

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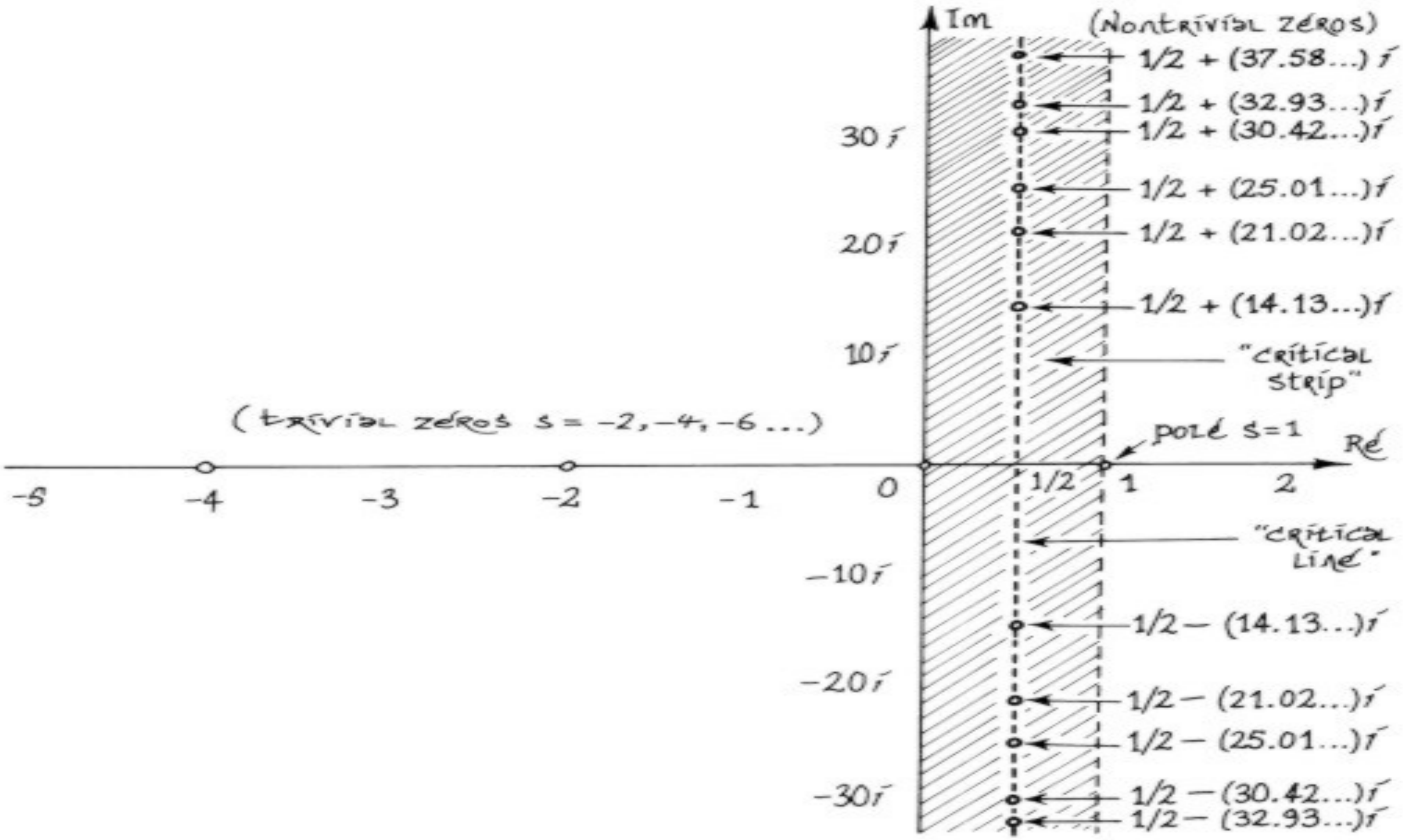
Quick look to the zeta function

$$\varsigma(s) = \sum_{1}^{\infty} \frac{1}{n^s}, \ \mathrm{Re}\, s > 1$$



The zeta function: Rosetta Stone in Maths

Zeta(s) can be written in three different "languages"

Sum over the integers (Euler)

$$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}\, s > 1$$

Product over primes (Euler)

$$\zeta(s) = \prod_{p=2,3,5,\ldots} \frac{1}{1-p^{-s}}, \mathrm{Re}\, s > 1$$

Product over the zeros (Riemann)

$$\zeta(s) \propto \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

The Riemann zeros and the prime numbers are dual objects

The RH imposes a bound to the chaotic behaviour of prime numbers

If RH is true then "there is music in the primes"
(M. Berry)

Brief history of the problem:

- In 1999 Berry and Keating proposed that the 1D Hamiltonian $H = x\, p$ could be related to the non trivial zeros of the Riemann zeta function.
- A quantization of xp may contain the Riemann zeros in the spectrum.
- This result would imply a proof of the Riemann hypothesis.
- The Berry-Keating proposal was parallel to Connes adelic approach where the Riemann zeros appear as missing spectral lines

Outline of the talk

- Polya and Hilbert conjecture and support
- H= xp: semiclassical approaches
- A self-adjoint extension of H = xp
- Russian doll BCS model of superconductivity
- H = xp + interactions: general theory
- Application to the Riemann zeros

Based on:
"The cyclic renormalization group and the Riemann zeros",
JSTAT (math.NT/0510572)

"H=xp with interaction and the Riemann zeros",
NPB (math-phys/0702034)

Polya and Hilbert conjecture (circa 1910)

The non trivial zeros of the zeta function are of the form

$$\varsigma(s) = 0, \quad s = \frac{1}{2} + i\, E$$

where E is an eigenvalue of a selfadjoint operator H and therefore a real number.

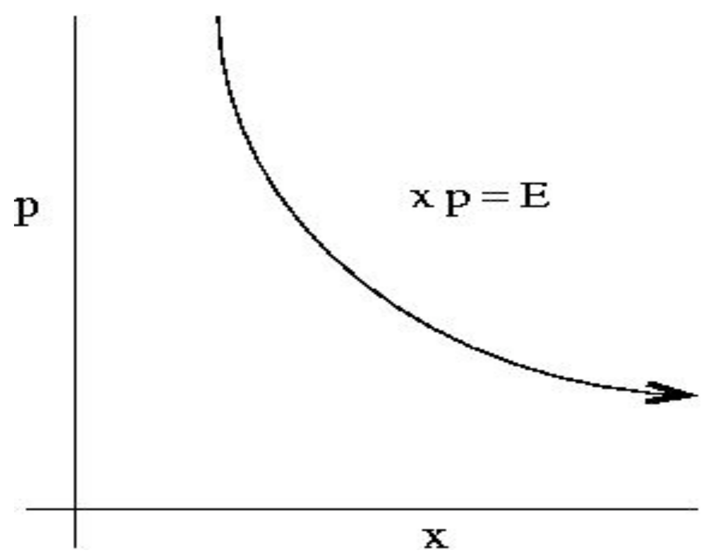
Montgomery-Odlyzko law: The zeros obey locally the Gaussian Unitary Ensemble (GUE) statistics which suggests that H breaks the time reversal symmetry.

Berry´s quantum chaos proposal: There is a classical chaotic system with isolated periodic orbits labelled by the prime numbers, which upon quantization yields the Riemann zeros in its spectrum.

Semiclassical approach to H = x p
Berry-Keating / Connes (1999)

Classically H = x p gives the trayectories

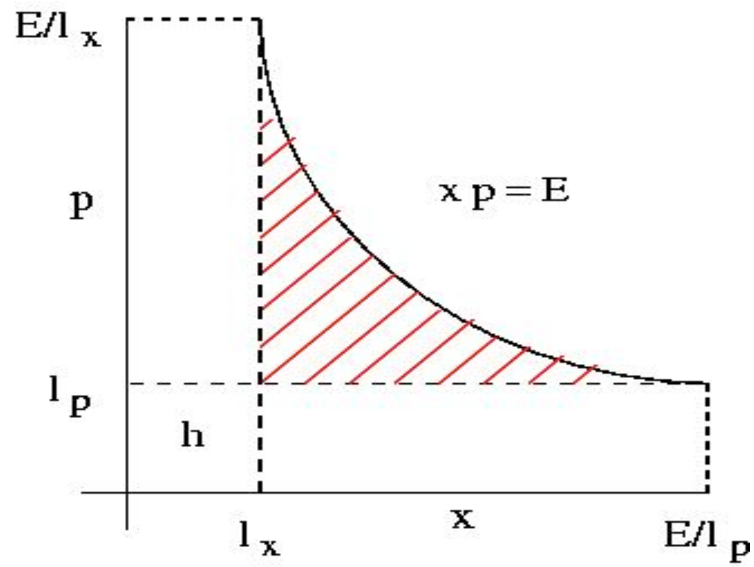
$$x(t) = x_0\, e^{t}\,,\, p(t) = p_0\, e^{-t}\,,\, E = x_0\, p_0$$



Time Reversal Symmetry is broken ($x \to x,\ p \to -p$)

Berry-Keating regularization

Planck cell in phase space: $|x| > l_x, \quad |p| > l_p, \quad h = l_x\, l_p = 2\pi, \; (\hbar = 1)$



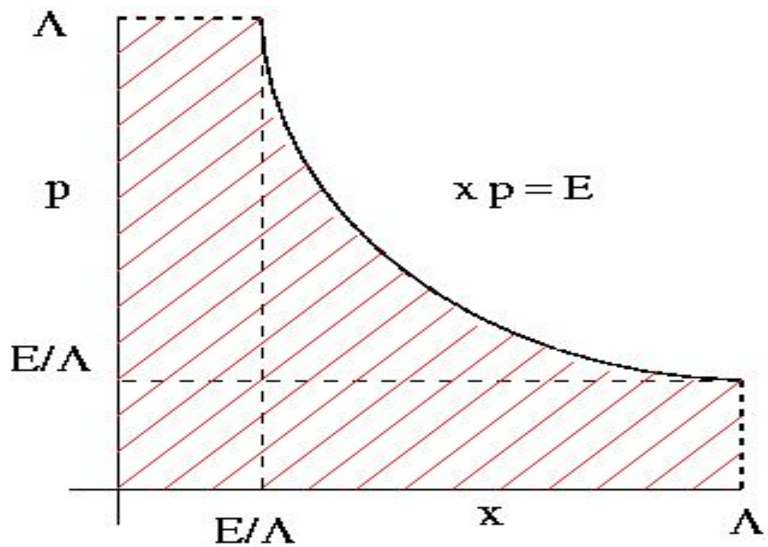
Berry–Keating cutoffs

Number of semiclassical states $N(E) = \frac{Area}{h} = \frac{E}{2\pi}\log\frac{E}{2\pi e} + 1$

Agrees with number of zeros asymptotically (smooth part)

Connes regularization

Cutoffs in phase space: $|x| < \Lambda, \quad |p| < \Lambda$

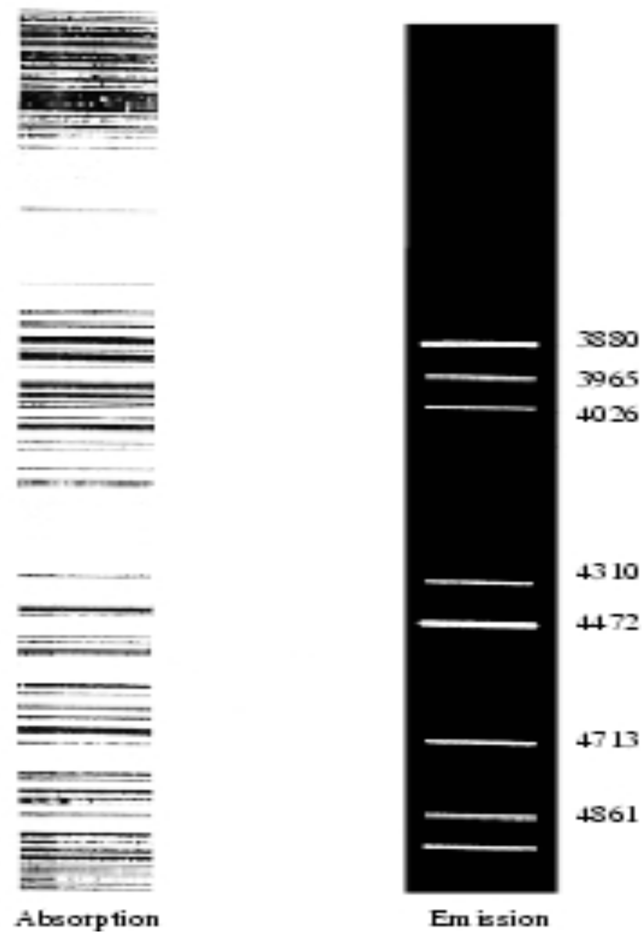


Connes cutoffs

Number of semiclassical states $N(E) = \frac{E}{\pi}\log\frac{\Lambda^2}{2\pi} - \frac{E}{2\pi}\log\frac{E}{2\pi e}$

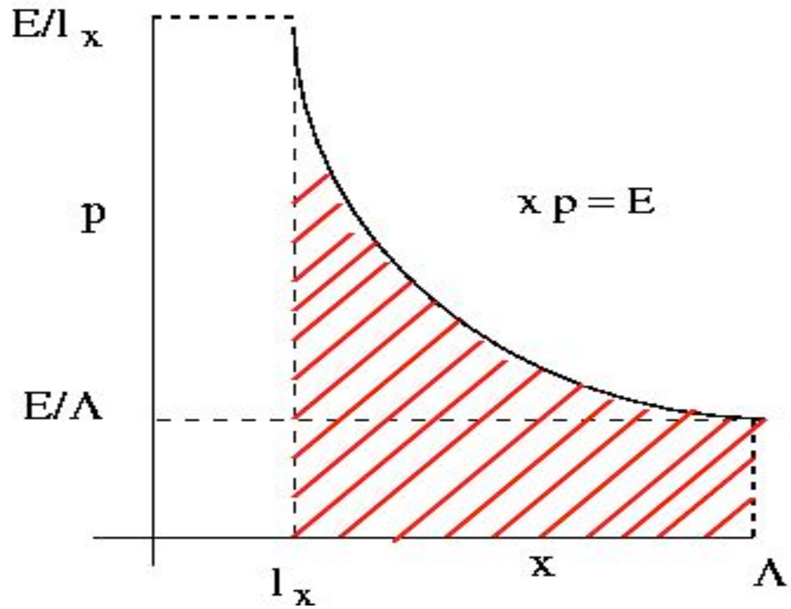
As $\Lambda \longrightarrow \infty$ spectrum = continuum - zeros

Spectral interpretation of the zeros: Absortion (Connes) or emission (Berry-Keating)?



A mixed regularization

Cutoffs in x space: $l_x < x < \Lambda$



Number of semiclassical states $N(E) = \frac{E}{2\pi} \log \frac{\Lambda}{l_x}$

There is a continuum of states as in Connes with no trace of the zeros unlike in Berry-Keating!!

A quantization of H = x p

Define the normal ordered operator

$$H = \frac{1}{2}(x\,p + p\,x) = -i\left(x\frac{d}{dx} + \frac{1}{2}\right)$$

The formal eigenfunctions are

$$\psi_E(x) = \frac{C}{x^{1/2 - i E}}$$

H admits a self-adjoint extension in the region 1 < x < N if

$$e^{i\theta}\,\psi(1) = N^{1/2}\,\psi(N)$$

which is satisfied by the eigenfunctions if

$$N^{iE} = e^{i\theta}$$

The spectrum is

$$E_n = \frac{2\pi}{\log N}\left(n + \frac{\theta}{2\pi}\right), \quad n = 0, \pm 1, \ldots$$

Which agrees with the semiclassical result

$$n = \frac{E_n}{2\pi}\log N - \frac{\theta}{2\pi}$$

For $\theta = \pi$ the spectrum is symmetric around zero

Relation with the Russian doll BCS model

The RD-BCS Hamiltonian (Leclair, Roman, GS, 2002)

$$H_{BCS}(x,x') = \varepsilon(x)\,\delta(x-x') - \frac{1}{2}\left(g + i\,h\; sign(x-x')\right)$$

$\varepsilon(x)$ = Energy levels, g = BCS coupling, h = T-breaking coupling

The many body model is exactly solvable (Dunning, Links)

For $\varepsilon(x) = x$ the eigenenergies and wave functions are

$$\left(\frac{N - E_{BCS}}{1 - E_{BCS}}\right)^{i\,h} = \frac{g + i\,h}{g - i\,h}, \qquad \phi_{BCS}(x) = \frac{C}{(x - E_{BCS})^{1 - i\,h}}$$

Map: BCS model and the H = xp model

$$E_{BCS} = 0 \;\; state \;\leftrightarrow\; E \neq 0 \;\; state$$

$$h \;\leftrightarrow\; E$$

$$h/g \;\leftrightarrow\; \tan(\theta/2)$$

$$\phi_{BCS}(x) \leftrightarrow x^{-1/2}\,\psi_{XP}(x)$$

The inverse Hamiltonian 1/xp

$$H = \frac{1}{2}(x\,p + p\,x) = x^{1/2}\,p\,x^{1/2} \to H^{-1} = x^{-1/2}\,p^{-1}\,x^{-1/2}$$

1/p is a 1D Green function, then

$$H^{-1}(x,x') = \frac{i}{2}\frac{sign(x-x')}{\sqrt{x\,x'}}$$

The Schroedinger equation is

$$\frac{i}{2}\int_1^N dx' \;\frac{sign(x-x')}{\sqrt{x\,x'}}\,\psi(x') = \frac{1}{E}\,\psi(x)$$

Which implies for $\phi(x) = x^{-1/2}\,\psi(x)$

$$x\,\phi(x) - \frac{i\,E}{2}\int_1^N dx'\;sign(x-x')\;\phi(x') = 0$$

This is the BCS model $\varepsilon(x) = x, \quad h = E, \quad g = 0 \to \theta = \pi$

H=x p + interactions: general model

Define the antisymmetric matrix

$$H^{-1}(x,x') = \frac{i}{2}\frac{sign(x-x') + a(x)b(x') - b(x)a(x')}{\sqrt{x\,x'}}, \quad 1 < x,x' < N$$

a(x) and b(x) are real “potentials”.
H(x,x´) is defined as the inverse of this matrix.
The interaction is a “proyector” into the states

$$\psi_a(x) = \frac{a(x)}{x^{1/2}}, \quad \psi_b(x) = \frac{b(x)}{x^{1/2}},$$

Type I: b(x)=1 and ψ_a is a square normalizable function.

Type II: ψ_a, ψ_b are square normalizable functions

The model is exactly solvable for generic potentials

Exact solution

The eigenenergies E are given by the solutions of

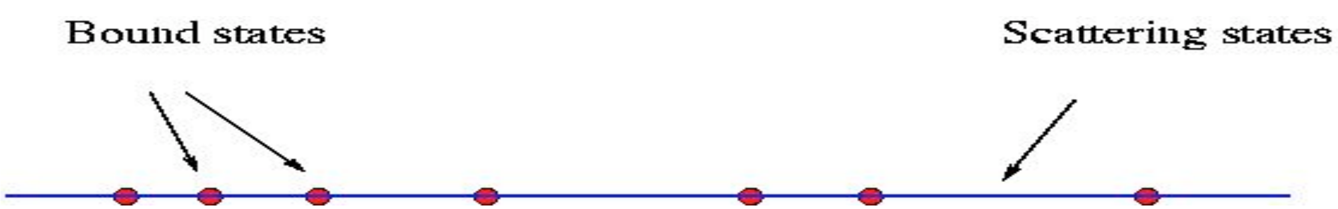
$$F(E)+F(-E)N^{iE}=0$$

F(E) plays the role of a Jost function

Spectrum in the limit $N \to \infty$

- If $F(E) \neq 0$ scattering state (delocalized)
- If $F(E) = 0$ bound state (localized)

The bound states (if any) are embbeded in the continuum!!



Similar to the von Neumann-Wigner potential (1 bound state with positive energy)

Eigenfunctions

$$\psi_E(x) = \frac{1}{x^{1/2-iE}}\left[C + \int_x^{\infty} dy\, y^{-iE}\left(A\frac{d\,b(y)}{dy} - B\frac{d\,a(y)}{dy}\right)\right]$$

A,B,C are integration constants which depend on E

$$C = F(E)$$

at large distances $\psi_E(x)$ is dominated by C term which vanishes when F(E) =0 and the state becomes localized

Localization is due to an interference effect and it is very sensitive to the form of the potential.

The Jost funtion F(E)

To find F(E) in the limit $N \to \infty$ use the Mellin transforms

$$\hat{f}(E) = \int_1^\infty dx\, x^{-1+iE}\, f(x), \qquad f = a, b$$

Given two functions f and g define

$$S_{f,g}(E) = -\frac{E^2}{4}\hat{f}(E)\,\hat{g}(-E) + \frac{iE}{4} P\int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{t\,\hat{f}(t)\,\hat{g}(-t)}{t-E}$$

Then

$$Type\ I: \quad F(E) = 1 + iE\,\hat{a}(E) - S_{a,a}(E)$$

$$Type\ II: \quad F(E) = 1 + S_{a,b} - S_{b,a} + S_{a,a}\, S_{b,b} - S_{a,b}\, S_{b,a}$$

F(E) is analytic in the complex upper half plane

Location of the zeros of F(E)

In the limit $N \to \infty$ the zeros are on the real axis or below it !!

$$If \quad F(E) = 0 \Rightarrow \operatorname{Im} E \leq 0$$

Proof: Since H is hermitean

$$\forall\, N, E, \operatorname{Im} E \neq 0 \Rightarrow F_N(E) + F_N(-E) N^{iE} \neq 0$$

Take $\operatorname{Im} E > 0, N \longrightarrow \infty \; then \quad F_\infty(E) \neq 0$

Real zeros are bound states

Complex zeros with $\operatorname{Im} E < 0$ are resonances

An example: the step potential

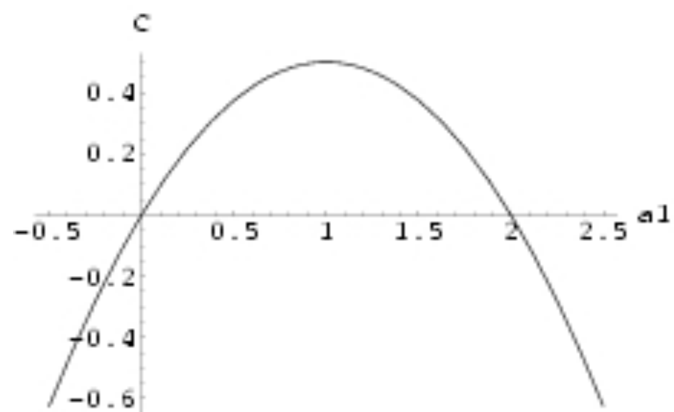
Take b(x) = 1 and $a(x) = \begin{cases} a_1 & 1 < x < x_1 \\ 0 & x_1 < x < \infty \end{cases}$

$$H^{-1}(x,x') = \frac{i}{2\sqrt{x\,x'}} \begin{pmatrix} 0 & -1 & a_1 - 1 & a_1 - 1 \\ 1 & 0 & a_1 - 1 & a_1 - 1 \\ 1 - a_1 & 1 - a_1 & 0 & -1 \\ 1 - a_1 & 1 - a_1 & 1 & 0 \end{pmatrix}$$

Jost function: $F(E) = 1 + C\,(x_1^{iE} - 1)$

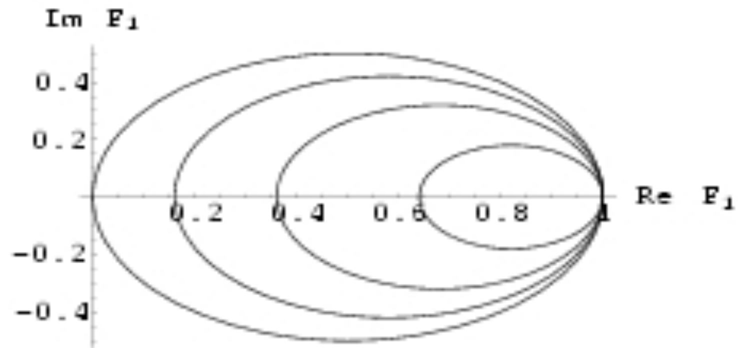
$$C = \frac{a_1(2 - a_1)}{2} \leq \frac{1}{2} \rightarrow a_1 = 1 \pm \sqrt{1 - 2C}$$

$$Symmetry: \quad a_1 \Leftrightarrow 2 - a_1$$



Argand plot of F(E) for

$a_1 = 0.2,\ 0.4,\ 0.6,\ 1$



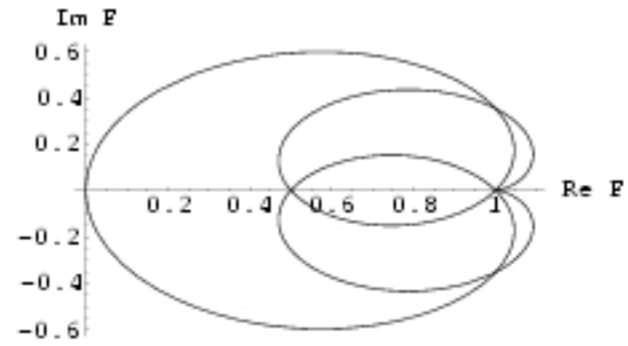
"Continuum" spectrum: $a_1 = 1 \longrightarrow E_n = \dfrac{2\pi}{\log(N/x_1)}(n+1/2),\ \ n = 0, \pm 1, \cdots$

Discrete spectrum: $a_1 = 1 \longrightarrow E_n = \dfrac{2\pi}{\log x_1}(n+1/2),\ \ n = 0, \pm 1, \cdots$

Zeros of F(E) $F(E) = 0 \Rightarrow |x_1^{iE}| = \left|1 + \dfrac{2}{a_1(a_1-2)}\right| \geq 1 \Rightarrow \operatorname{Im} E \leq 0$

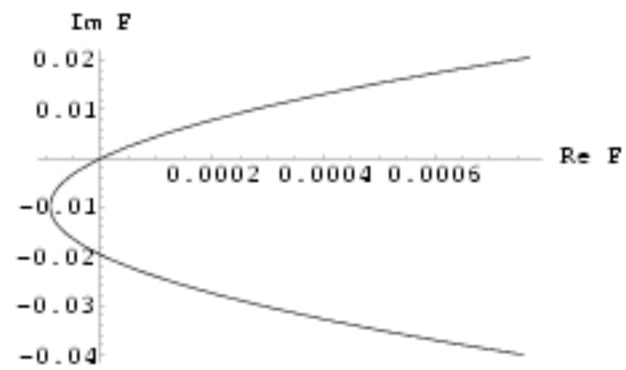
Two step potentials

$$a(x) = a_1\,\theta(x_1 - x), \quad b(x) = b_1\,\theta(x_2 - x)$$



Algebraic potentials

$$a(x) = \frac{a_1}{x} + \frac{a_2}{x^4}, \quad b(x) = \frac{b_1}{x^{0.03}}$$

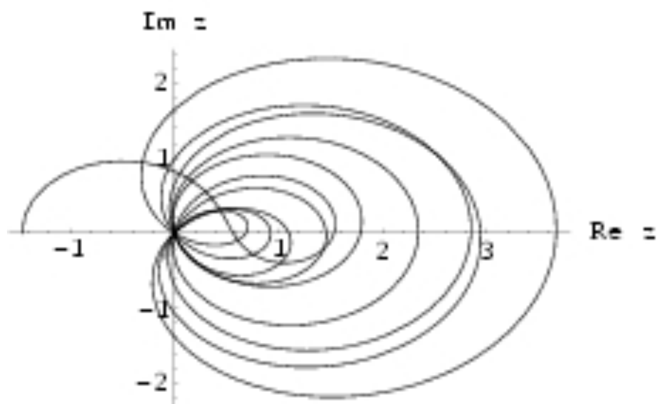


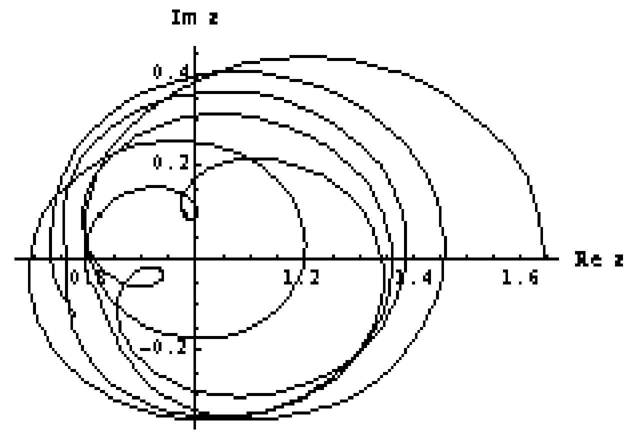
$$Type\ I: \quad \mathrm{Re}\,F(E) = \left|1 + \frac{i}{2} E\,\hat{a}(E)\right|^2 \geq 0$$

$$Type\ II: \quad \mathrm{Re}\,F(E) \geq 0 \ \ or \ \ \leq 0$$

Application to the zeta function

Argand plot of $\zeta(1/2 - i\,t)$ *and* $\zeta(2 - i\,t)$





Suggestion: $\zeta(\sigma - i\,t)$ is proportional to the Jost function for a model of type I ($\sigma > 1$) or type II ($1/2 \leq \sigma \leq 1$)

1) Exact perturbative potential a(x) for $\zeta(\sigma - i\,t), \quad \sigma > 1$

2) Approximate potential a(x) for the smooth Riemann zeros

3) Approximate potentials a(x), b(x) for $\zeta(1/2 - i\,t)$

Perturbative solution of the model I

$$F(t) = 1 - 4a + 4a * a, \qquad a(t) = -it\hat{a}(t)/4$$

Where the * product is defined as

$$(a * b)(z) = a(z)b(-z) + P\int_{-\infty}^{\infty} \frac{dt}{i\pi} \frac{a(t)b(-t)}{t-z}$$

Writting $a = g + a * a, \quad g = (1-F)/4$ and iterating one gets

$$a = g + g * g + (g * g) * g + g * (g * g) + \ldots$$

There are $C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1} \approx \frac{4^n}{n^{3/2}}$ terms to the power g^n

The series converges if $|F(t) - 1| \leq 1, \quad \forall t$

The Riemann zeta function with $\sigma > 1$

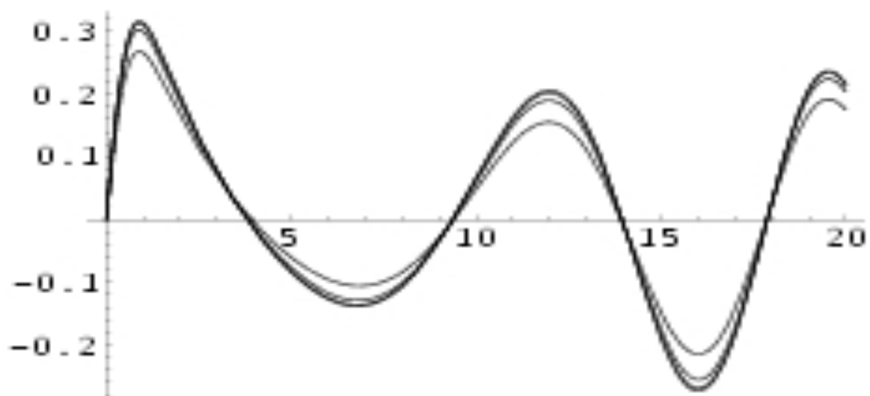
$$F(t) = C\,\varsigma(\sigma - i\,t), \quad \sigma > 1, \quad C = 1/\varsigma(\sigma)$$

It satisfies $F(0) = 1, \quad \mathrm{Re}\, F(t) \geq 0, \quad |F(t) - 1| \leq 1, \quad \forall t$

$$a(t) = c_1 + c_2\,\varsigma(\sigma - i\,t) + c_3\,\varsigma_2(\sigma, t) + \ldots$$

Where $\varsigma_2(\sigma,t) = \sum_{n>m\geq 0} \frac{1}{n^{\sigma - it}\; m^{\sigma + it}}$ is a Euler-Zagier zeta function

For $\sigma = 2$



Division versus multiplication

The Jost function can be written as

$$F = f * f, \quad f = 1 - 2a$$

The * product implements the division of numbers

$$f = c_1 r_1^{it} + c_2 r_2^{it} \rightarrow f * f = c_1^2 + c_2^2 + 2 c_1 c_2 \left(\frac{r_1}{r_2}\right)^{it}, \quad (r_1 > r_2)$$

Hence the *square root of zeta(s) encodes all rational numbers

$$\varsigma = f * f \rightarrow Integers = \sum \frac{Rational}{Rational}$$

$$\varsigma = \sum_{n \geq 1} \frac{n^{it}}{n^{\sigma}} \; (n \in \mathrm{N}) \qquad f = \sum_{r \geq 1} c_r r^{it} \; (r \in Q)$$

To be compare with the Euler product formula

$$\varsigma = \prod_p \frac{1}{1 - p^{-s}} \; (p \in P) \rightarrow Integer = \Pi \; primes$$

Approximate potential a(x) for the smooth Riemann zeros

Zeta function on the critical line $\varsigma(1/2 - i\,t) = Z(t)\,e^{i\vartheta(t)}$

Z(t) :Riemann-Siegel zeta function

$\vartheta(t)$ is a phase

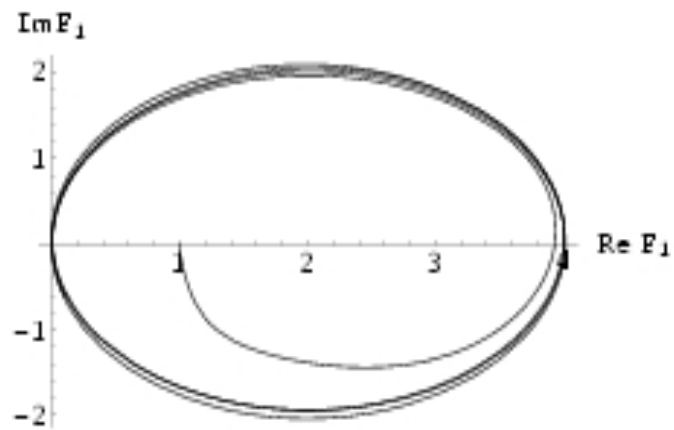
$$e^{2i\vartheta(t)} = \pi^{-it}\,\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}$$

Zeros of Z(t) = 0 are close to the zeros of $1 + e^{2i\vartheta(t)} = 0$

Choose the potential $a(x) = -2\frac{\sin(2\pi x)}{\sqrt{x}} \rightarrow \hat{a}(t) = -\frac{2i}{t}e^{2i\vartheta(t)} + O(t^{-2})$

The Jost function is:

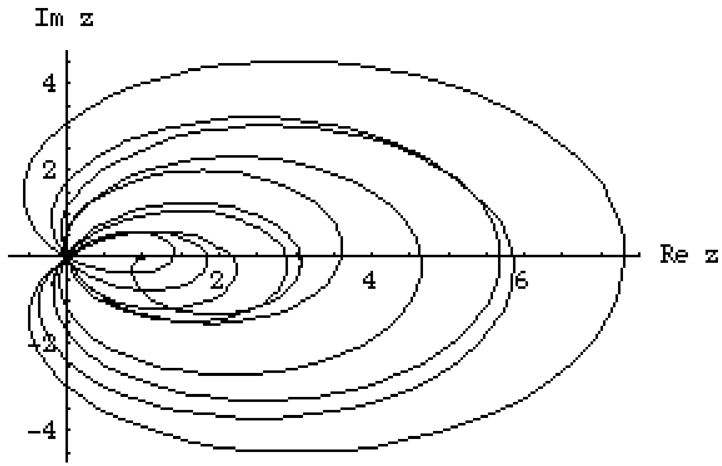
$$F(E) = 2\left(1 + e^{2i\vartheta(t)}\right) + O(t^{-1})$$



Smooth zeros are quasibound states

Approximate potentials a(x), b(x) for $\sigma = 1/2$ (work in progress)

$$F(t) = C\frac{t - i/2}{t + i\mu}\varsigma(1/2 - it)$$



The potentials can be choosen such that

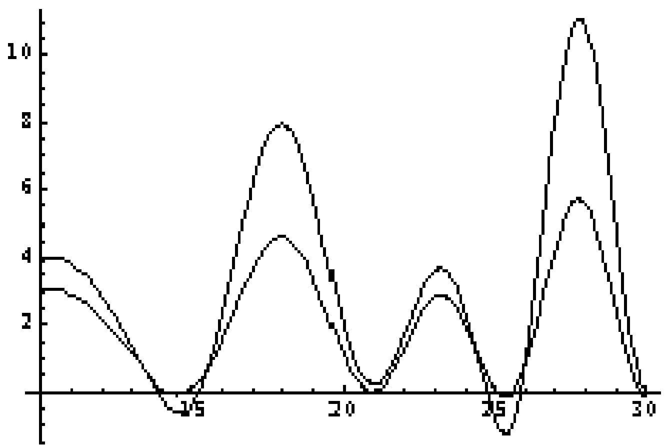
$$F = 1 - 2ab^{*} + (a * a)(b * b)$$

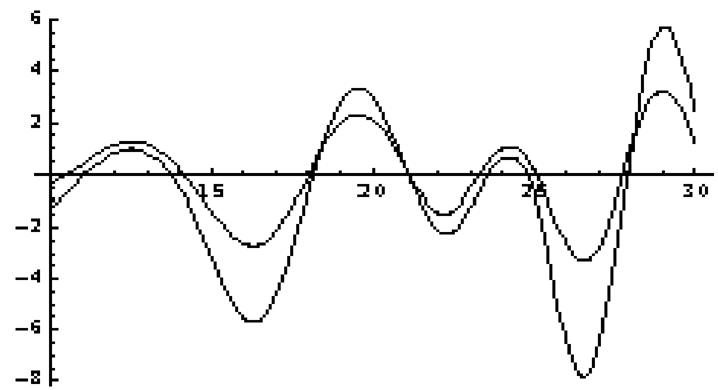
Make the guess $|a(t)| = |b(t)|$

$$F = 1 - 2c + c \otimes c, \qquad c(t) = a(t)b(-t)$$

$$(c \otimes c)(z) = \left(|c(z)| + \int_{-\infty}^{\infty} \frac{dt}{i\pi} \frac{|c(t)|}{t - z} \right)^2 = (|c(z)| - i\kappa(z))^2$$

Choosing $c(t) = (1 - F(t))/2$





Gives a good approximation near the zeros

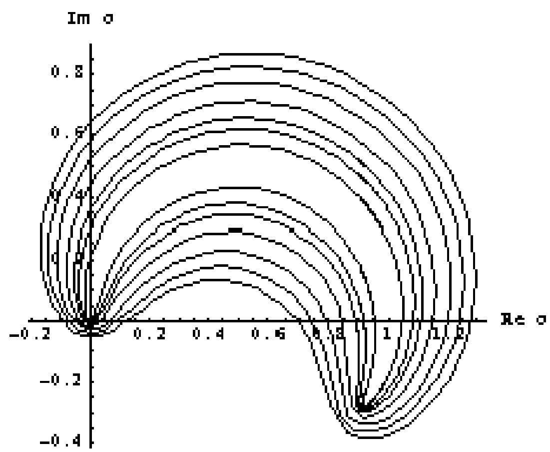
Real and imaginary parts

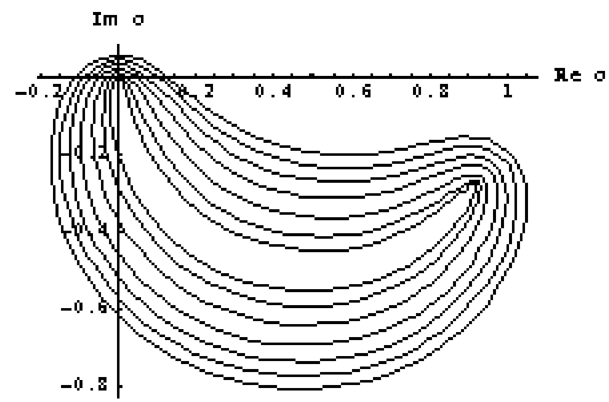
$$F_R = 1 - 2c_R + |c|^2 - \kappa^2$$
$$F_I = -2(c_I + |c|\kappa)$$

Eliminating κ gives the generic locus of the potential

$$(c_R^2 + c_I^2)(c_R^2 + c_I^2 - 2c_R + 1 - F_R) - (c_I + F_I/2)^2 = 0$$

Before and after the first Riemann zero t=14.1342…





At the zeros of zeta(1/2 - i t) the potential is:

$$c(t_0) = \sin\vartheta_0 \; e^{i(\vartheta_0 - \pi/2)} \qquad (\vartheta_0 = \vartheta(t_0))$$

Away from the zeros one expects $\quad |a(t)| \neq |b(t)|$

Conclusions and suggestions

- Consistent quantization of H = xp, related to a BCS model where time reversal is broken
- H = xp + interaction model is a promising candidate to realize the Polya-Hilbert conjecture.
- If correct the Riemann zeros would be discrete spectral lines embbeded in the continuum, reconciling the Berry-Keating and Connes spectral interpretations.
- H = x p can be generalized to discrete models (universality).
- The Dirichlet L functions can be obtained at the smooth approximation.

Much more work is needed to answer these questions.
The story will continue….