Stretched States from Majorana Modes

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G.Semenoff and P.Sodano **cond-mat/0601261 /0605147 /0707???**

Outline:

- Fermion zero modes and fractional charge
- Majorana fermion zero modes
  - $(-1)^F$ and hidden variables
- Majorana fermions in a soliton-antisoliton system
  - Teleportation
- Spin degeneracy
  - Majorana zero modes as a qubit

Fractionally charged soliton

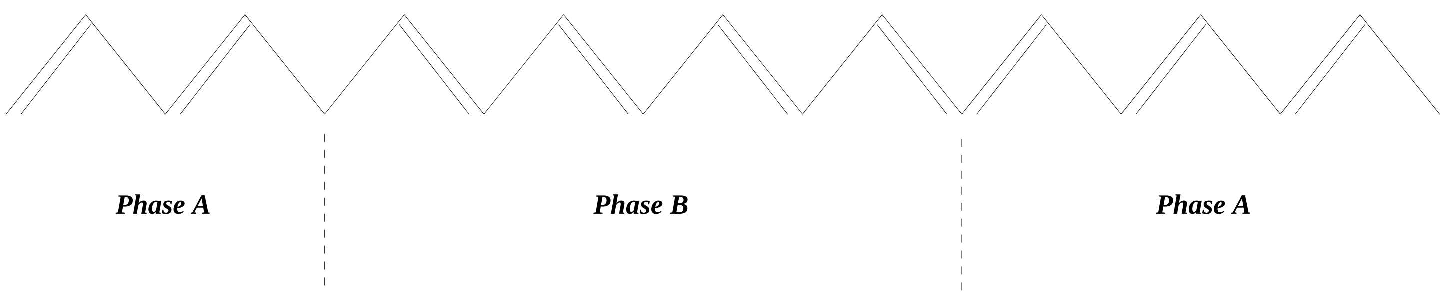
R.Jackiw, C.Rebbi, Phys.Rev.D13,3398 (1976)

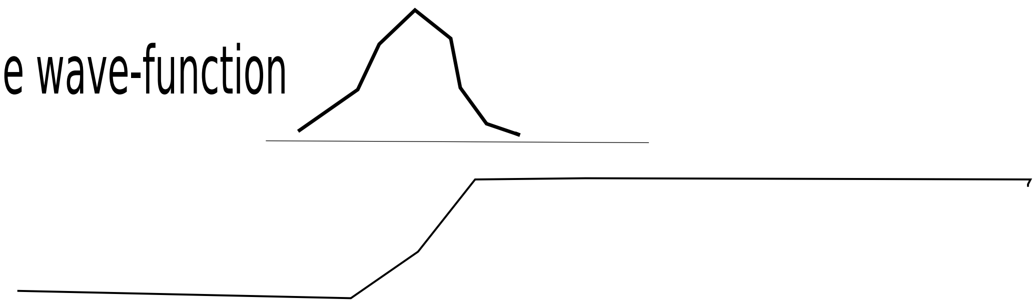
W.P.Su, J.R.Schrieffer, A.Heeger, Phys.Rev.Lett.42,1698(1979)

R.Jackiw, J.R.Schreiffer, Nucl.Phys.B190[FS3],253(1981).

Fermion interacting with a topological solition can have quantum states with fractional charge $q = \left(n + \frac{1}{2}\right) e$ (also can be non-($\frac{1}{2}$)integer in systems with less symmetry)

Example in 1+1-dimensions which models polyacetylene





Scalar field with soliton profile: $\phi(x) = \mu \tanh \mu x$

Dirac Eqn. in soliton background: $(i\gamma \cdot \partial - \phi(x))\,\psi(x,t) = 0$

$\psi(x,t) = e^{iEt}\psi_E(x)$

Hamiltonian $\left(i\vec{\alpha} \cdot \vec{\nabla} + \beta\phi(x)\right)\psi_E(x) = E\psi_E(x)$

$$\begin{bmatrix} 0 & -\partial_x + \phi(x) \\ \partial_x + \phi(x) & 0 \end{bmatrix} \begin{bmatrix} u_E(x) \\ v_E(x) \end{bmatrix} = E \begin{bmatrix} u_E(x) \\ v_E(x) \end{bmatrix} \ , \quad \begin{bmatrix} u_{-E} \\ v_{-E} \end{bmatrix} = \begin{bmatrix} u_E \\ -v_E \end{bmatrix}$$

$\exists$ zero frequency $E = 0$ mode. $u_0(x) \sim e^{-\int_0^x \phi(x')dx'}$, $v_0 = 0$

$$\psi(x,t) = \psi_0(x)a + \sum_{E>0} \left[e^{iEt}\psi_E(x)a_E + e^{-iEt}\psi_{-E}(x)b_E^\dagger\right]$$

- Charge conjugation symmetry

$$\psi(x,t) \to C\psi^{\dagger}(c,t) \quad , \quad \psi_E(x) = C\psi^{*}_{-E}(x) \quad , \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Q \to CQC^{-1} = -Q$$

For every state $|q>$ with $Q|q>= q|q>$, $\exists \ \ |-q>= C|q>$ with $Q|-q>= -q|-q>$

- charges are quantized: if $\exists \ \ |q>$ with $Q|q>= q|q>$ then $\exists \ \ |q+e>$ with $Q|q+e>= (q+e)|Q+e>$

Then $q-(-q) = 2q = e\cdot$integer and **either**

- $q = n \cdot e \qquad n \in \mathcal{Z}$
  or
- $q = \left(n + \frac{1}{2}\right) \cdot e \qquad n \in \mathcal{Z}$

$$\psi(x,t)=\psi_0(x)a+\sum_{E>0}\left[e^{iEt}\psi_E(x)a_E+e^{-iEt}\psi_{-E}(x)b_E^\dagger\right]$$

Zero mode creation/annihlation operator

$$\{a,a^\dagger\}=1\ ,\quad \{a,a_E\}=0=\{a,b_E\}\quad ,\quad \{a_E,a_{E'}^\dagger\}=\delta_{EE'}\quad \text{etc.}$$

Zero mode has two-dimensional representation $\{|\downarrow>,|\uparrow>\}$

$$a^\dagger|\downarrow>=|\uparrow>\quad ,\ a^\dagger|\uparrow>=0\ ,\ a|\uparrow>=|\downarrow>\quad ,\ a|\downarrow>=0$$

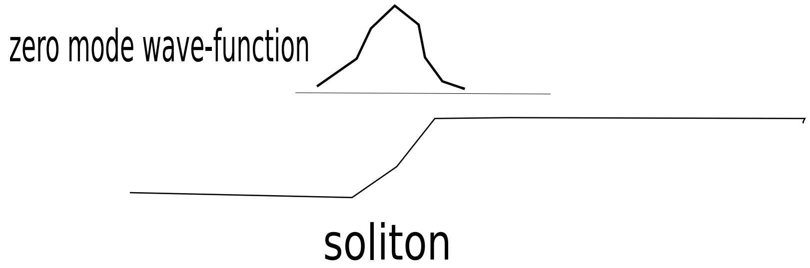
$a_E|\uparrow>=a_E|\downarrow>=0=b_E|\uparrow>=b_E|\downarrow>$
$a_{E_1}^\dagger...b_{E_1'}^\dagger...|\uparrow>\ ,\ a_{E_1}^\dagger...b_{E_1'}^\dagger...|\downarrow>$ have energy $E_1+...+E_1''+...$

$$Q/e=\int dx\psi^\dagger(x,t)\psi(x,t)-\text{const}=a^\dagger a-\frac{1}{2}+\sum_E\left(a_E^\dagger a_E-b_E^\dagger b_E\right)$$

$$Q|\downarrow>=-\frac{e}{2}|\downarrow>\quad ,\quad Q|\uparrow>=\frac{e}{2}|\uparrow>\quad ,\quad Q=\left(n+\frac{1}{2}\right)e$$

Majorana fermion: identify particles and anti-particles

$\psi_{-E}(x) = C\psi_E^*(x) \rightarrow$ impose $\psi(x,t) = C\psi^*(x,t)$



$$\psi(x,t) = \psi_0(x)a + \sum_{E>0}\left[e^{iEt}\psi_E(x)a_E + e^{-iEt}\psi_{-E}(x)a_E^\dagger\right]$$

$$a = a^\dagger \ , \ a^2 = \frac{1}{2} \quad , \quad \{a, a_E\} = 0 \quad , \quad \{a_E, a_{E'}^\dagger\} = \delta_{EE'}$$

Zero mode has 2 inequiv. 1-dim.reps.

$$a = (\pm)\frac{1}{\sqrt{2}}(-1)^{\sum_E a_E^\dagger a_E} \quad , \quad a_E|0>= 0 \quad , \quad a|0>= (\pm)\frac{1}{\sqrt{2}}|0>$$

States: $a_{E_1}^\dagger ...|0>$ energy $E_1 + ...$

Majorana fermion: identify particles and anti-particles

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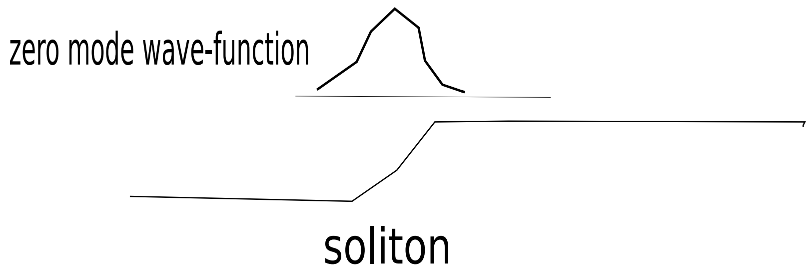

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Zero mode has 2 inequiv. 1-dim.reps.

$$a = (\pm)\frac{1}{\sqrt{2}}(-1)^{\sum a_E^\dagger a_E} \;\; , \;\; a_E|0>= 0 \;\; , \;\; a|0>= (\pm)\frac{1}{\sqrt{2}}|0>$$

States: $a_{E_1}^\dagger ...|0>$ energy $E_1 + ...$ **but** $<0|\psi(x,t)|0>= (\pm)\frac{1}{\sqrt{2}}\psi_0(x)$

Majorana fermions do not have conserved fermion number:
$(Q = e\int \psi^\dagger\psi = 0)$

They do have a discrete symmetry, $(-1)^F\psi(x,t) + \psi(x,t)(-1)^F = 0$
(**fermion parity**: conservation of fermion number mod 2)

$$\left[(-1)^F, \psi^\dagger M\psi\right] = 0$$

However, since $a|0>=(\pm)\frac{1}{\sqrt{2}}|0>$, $|0>$ not an eigenstate of $(-1)^F$

$-----------------------------------$

Fixed by using 2-dim. rep of algebra (**hidden variable**)

$$a^2 = \frac{1}{2}\ ,\ b^2 = \frac{1}{2}\ ,\ \{a,b\} = 0\ ,\ \alpha = a + ib\ ,\ \alpha^\dagger = a - ib$$

$\{\alpha,\alpha^\dagger\} = 1\ ,\ \alpha|->=0\ ,\ \alpha^\dagger|->=|+>\ ,\ \alpha^\dagger|+>=0\ ,\ \alpha^\dagger|+>=|->$

Fermion parity: $(-1)^F|->=-|->\ ,\ (-1)^F|+>=|+>$

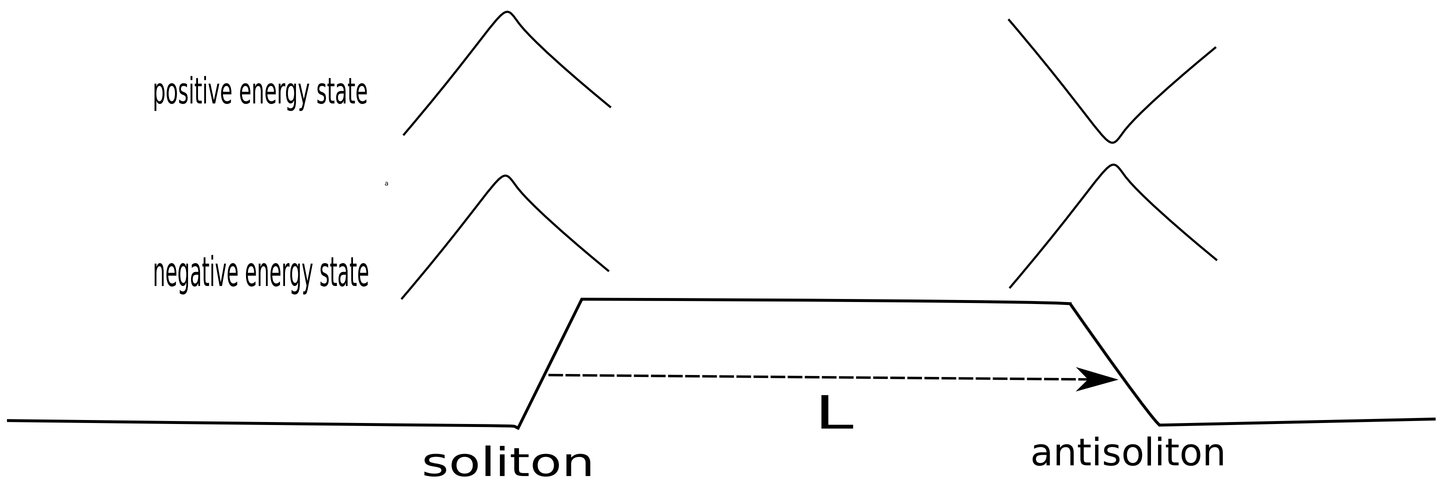
Soliton-antisoliton system $-->$

Fractional charge in a two-soliton system

R.Jackiw, A.Kerman, I.Klebanov, G.Semenoff,

Nucl.Phys.B225 [FS9], 233 (1983)

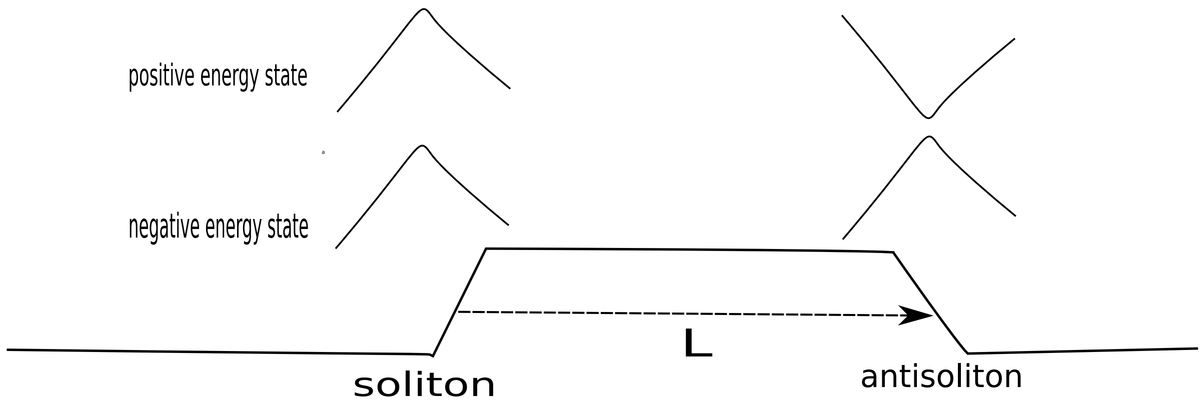
R.Jackiw J.R.Schrieffer, cond-mat/0012370



Two bound states with $\omega = \pm\epsilon \sim e^{-L}$ $\quad L \to \infty$

$$\psi = \psi_+(x)e^{i\epsilon t}\alpha + \psi_-(x)e^{-i\epsilon t}\beta^\dagger + \sum_{E>0}\left[e^{iEt}\psi_E(x)\alpha_E + e^{-iEt}\psi_{-E}(x)\beta_E^\dagger\right]$$

$$= \left[\frac{\psi_+ e^{i\epsilon t} + \psi_- e^{-i\epsilon t}}{\sqrt{2}}\right]\frac{\alpha+\beta^\dagger}{\sqrt{2}} + \left[\frac{\psi_+ e^{i\epsilon t} - \psi_- e^{-i\epsilon t}}{-\sqrt{2}i}\right]\frac{\alpha-\beta^\dagger}{\sqrt{2}i} + \ldots$$

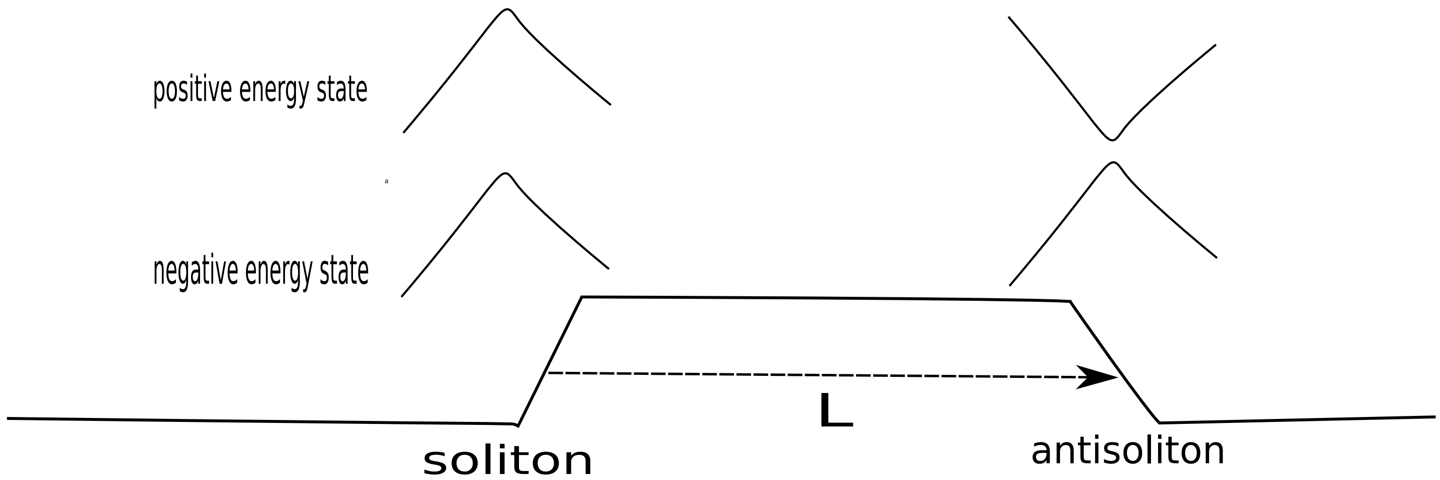


$$\psi = \left[\frac{\psi_+ e^{i\epsilon t} + \psi_- e^{-i\epsilon t}}{\sqrt{2}}\right] \frac{\alpha + \beta^\dagger}{\sqrt{2}} + \left[\frac{\psi_+ e^{i\epsilon t} - \psi_- e^{-i\epsilon t}}{-\sqrt{2} i}\right] \frac{\alpha - \beta^\dagger}{\sqrt{2} i} + \ldots$$

$$\sim \begin{bmatrix} u_0 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ v_0 \end{bmatrix} b + \ldots$$

$$a = \frac{\alpha + \beta^\dagger}{\sqrt{2}} \ , \ a^\dagger = \frac{\alpha^\dagger + \beta}{\sqrt{2}} \ , \ \{a, a^\dagger\} = 1$$

$$b = \frac{\alpha - \beta^\dagger}{\sqrt{2} i} \ , \ b^\dagger = \frac{\alpha^\dagger - \beta}{-\sqrt{2} i} \ , \ \{b, b^\dagger\} = 1$$



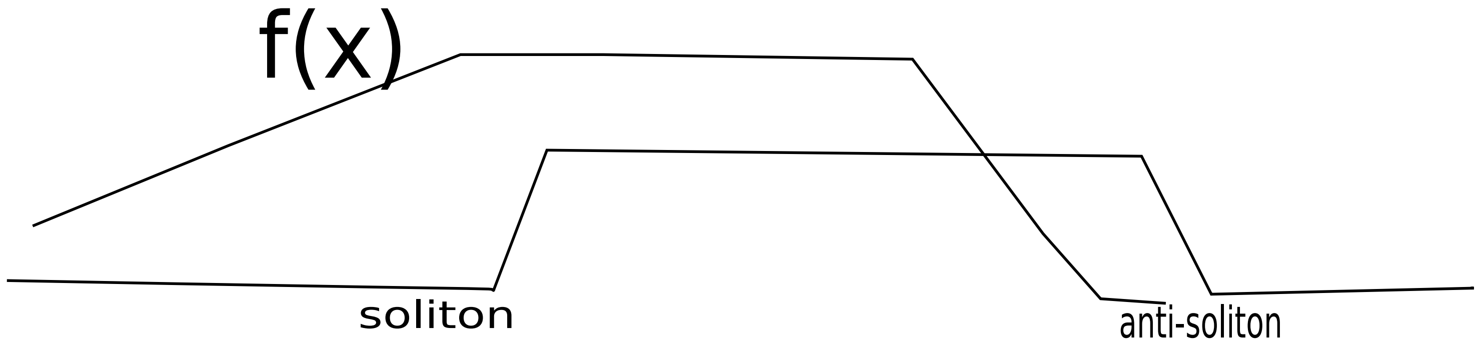
$$a = \frac{\alpha + \beta^\dagger}{\sqrt{2}} \ , \ a^\dagger = \frac{\alpha^\dagger + \beta}{\sqrt{2}} \ , \ \{a, a^\dagger\} = 1$$

$$b = \frac{\alpha - \beta^\dagger}{\sqrt{2}i} \ , \ b^\dagger = \frac{\alpha^\dagger - \beta}{-\sqrt{2}i} \ , \ \{b, b^\dagger\} = 1$$

has a four-dimensional representation

$$a|\downarrow>= 0 \ , \ a^\dagger|\downarrow>= |\uparrow> \ , \ a|\uparrow>= |\downarrow> \ , \ a^\dagger|\uparrow>= 0$$

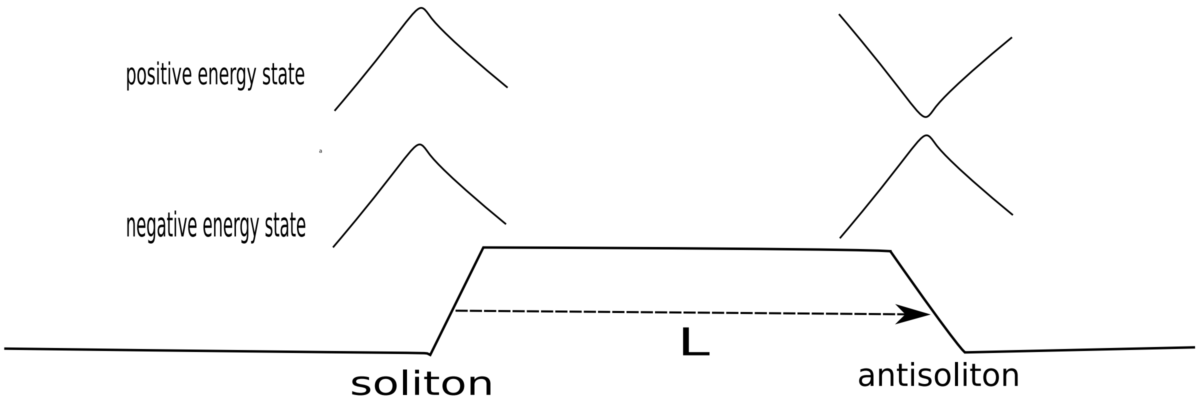
$$b|\downarrow>= 0 \ , \ b^\dagger|\downarrow>= |\uparrow> \ , \ b|\uparrow>= |\downarrow> \ , \ b^\dagger|\uparrow>= 0$$



States are direct products $\begin{array}{cc} |\uparrow\uparrow> & |\uparrow\downarrow> \\ |\downarrow\uparrow> & |\downarrow\downarrow> \end{array}$ with fractional charge.

$$Q = \lim_{f\to 1}\lim_{L\to\infty} e\int dx f(x)\psi^\dagger(x,t)\psi(x,t) \quad , \quad \begin{array}{c} Q|\uparrow\uparrow>=\frac{e}{2}|\uparrow\uparrow> \\ Q|\downarrow\uparrow>=-\frac{e}{2}|\downarrow\uparrow> \end{array}$$

With Majorana fermions



$$\psi = \psi_+(x)e^{i\epsilon t}\alpha + \psi_-(x)e^{-i\epsilon t}\alpha^\dagger + \sum_{E>0}\left[e^{iEt}\psi_E(x)\alpha_E + e^{-iEt}\psi_{-E}(x)\alpha_E^\dagger\right]$$

$$= \frac{\psi_+ e^{i\epsilon t} + \psi_- e^{-i\epsilon t}}{\sqrt{2}}\frac{\alpha+\alpha^\dagger}{\sqrt{2}} + \frac{\psi_+ e^{i\epsilon t} - \psi_- e^{-i\epsilon t}}{-\sqrt{2}i}\frac{\alpha-\alpha^\dagger}{\sqrt{2}i} + \ldots$$

$$\alpha|- >= 0\ ,\ \alpha^\dagger|- >= |+ >\ ,\ \alpha^\dagger|+ >= 0\ ,\ \alpha|+ >= |- >$$

'local variables': $a = \frac{\alpha+\alpha^\dagger}{\sqrt{2}} = a^\dagger\ ,\ a^2 = \frac{1}{2}\ ;\ b = \frac{\alpha-\alpha^\dagger}{\sqrt{2}i} = b^\dagger\ ,\ b^2 = \frac{1}{2}$

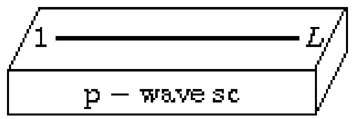
Superpositions of $|+ >, |- >$ violate $(-1)^F$

classical switch off$= |- >$ **on**$= |+ >$ (stretched states)

Position of switch measured by $iab|+ >= +|+ >$, $iab|- >= -|- >$

quantum wire + $p_x + ip_y$-superconductor

A.Kitaev cond-mat/0010440



$$\left\{\psi_k, \psi^\dagger_{k'}\right\} = \delta_{kk'}$$

$$H = \sum_{k=1}^{L} \left\{ \frac{t}{2} \left[\psi^\dagger_{k+1}\psi_k + \psi^\dagger_k \psi_{k+1}\right] + \mu \psi^\dagger_k \psi_k + \frac{\Delta}{2}\psi^\dagger_{k+1}\psi^\dagger_k + \frac{\Delta^*}{2}\psi_k \psi_{k+1} \right\}$$

t= amplitude for electron hopping between neighboring sites

Δ= amplitude for electrons leave wire as cooper pair

p-wave condensate $\sim < \psi_\sigma \vec{r} \times \vec{\nabla}\psi_\sigma >$, consider one spin state

Choose phases so that Δ, t are real, positive. $t > \Delta > \mu$

Real and imaginary parts of electron $\psi_k(t) = b_k(t) + ic_k(t)$

Schrödinger equation:

$$i\frac{d}{dt}\begin{bmatrix} b_k \\ c_k \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{bmatrix}\begin{bmatrix} b_k \\ c_k \end{bmatrix} \quad , \quad \begin{bmatrix} b_0 \\ c_0 \end{bmatrix} = 0 = \begin{bmatrix} b_{L+1} \\ c_{L+1} \end{bmatrix}$$

$$\mathcal{D}b_k = i\left\{\frac{t}{2}\left(b_{k+1} + b_{k-1}\right) + \mu b_k + \frac{\Delta}{2}\left(b_{k+1} - b_{k-1}\right)\right\}$$

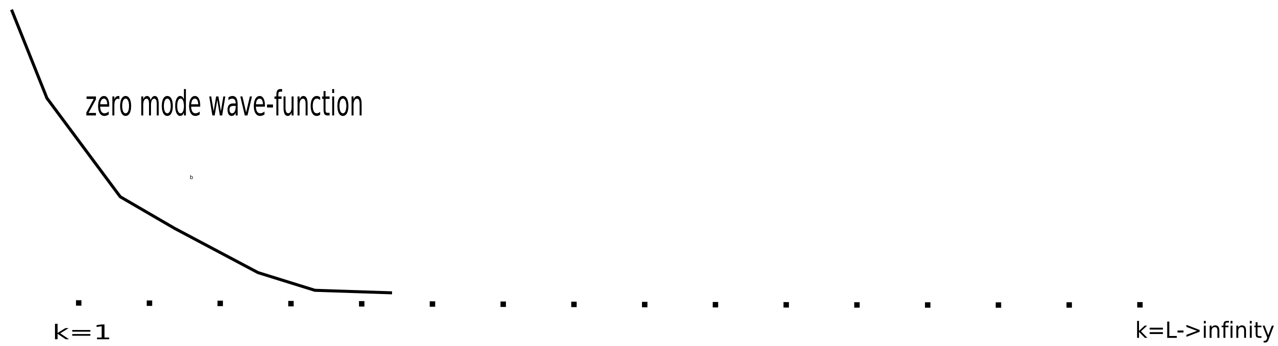
$$\mathcal{D}^\dagger c_k = -i\left\{\frac{t}{2}\left(c_{k+1} + c_{k-1}\right) + \mu c_k - \frac{\Delta}{2}\left(c_{k+1} - c_{k-1}\right)\right\}$$

1-1 mapping of positive to negative energy levels $\begin{bmatrix} b \\ c \end{bmatrix} \to \begin{bmatrix} b \\ -c \end{bmatrix}$

# of zero modes mod2 is a topological invariant

Can be explicitly solved for the half-line: $(L \to \infty)$ $\quad c_k^0 = 0$

$$b_k^0 = 2\sqrt{\frac{t^2 \sin^2\phi + \Delta^2 \cos^2\phi}{(t^2 - \Delta^2)\sin^2\phi}}\left(\frac{t-\Delta}{t+\Delta}\right)^k \sin k\phi\,,\ \phi = \cos^{-1}\frac{\mu}{\sqrt{t^2 - \Delta^2}}$$



All other modes are in the continuum with

$$E(p) = \sqrt{(t\cos p + \mu)^2 + \Delta^2 \sin^2 p}$$

$$\psi_k(t) = b_k^0 \beta + \int_0^\infty dp \left((b_k(p) + ic_k(p))\beta_p e^{iE(p)t} + (b_k^*(p) - ic_k^*(p))e^{-iE(p)t}\gamma_p^\dagger \right)$$

$$\beta = \beta^\dagger \quad . \quad \beta^2 = \frac{1}{2} \quad , \quad \{\beta, \beta_p\} = 0 \quad , \quad \{\beta, \gamma_p\} = 0$$

- When L is finite, there are two bound states with energies $\pm\epsilon \sim e^{-L}$.
- b_k^0 is localized near $k = 1$ and, up to exponentially small corrections, is the same as in the half-line. $c_k^0 = b_{L+1-k}^0$.
- The negative energy bound state has wave-function $(b_k^0 + ic_k^0)e^{-i\epsilon t}$ and the positive energy state is $(b_k^0 - ic_k^0)e^{i\epsilon t}$.

$$\psi_k(t) = (b_k^0 + ib_{L+1-k}^0)e^{i\epsilon t}\beta + (b_k^0 - ib_{L+1-k}^0)e^{-i\epsilon t}\beta^\dagger + \ldots$$

$$= b_k^0 \; a \;\; + \;\; ib_{L+1-k}^0 \; b \;\; + \ldots$$

$$a = \frac{\beta + \beta^\dagger}{2} \;, \; a = a^\dagger \;, \; a^2 = \frac{1}{2} \;, \; b = \frac{\beta - \beta^\dagger}{2i} \;, \; b = b^\dagger \;, \; b^2 = \frac{1}{2}$$

$$a = \frac{\beta + \beta^\dagger}{2} \ , \ a = a^\dagger \ , \ a^2 = \frac{1}{2} \ , \ b = \frac{\beta - \beta^\dagger}{2i} \ , \ b = b^\dagger \ , \ b^2 = \frac{1}{2}$$

stretched states

$$\beta|- >= 0 \ , \ \beta^\dagger|- >= |+ > \ , \ \beta|+ >= |- > \ , \ \beta|- >= 0$$

Teleportation

G.Semenoff and P.Sodano cond-mat/0601261 /0605147
S.Tewari,C.Zhang,S.DasSarma,C.Nayak,D.Lee
cont-mat/0703717

$$H = E\chi_1^\dagger\chi_1 + E\chi_2^\dagger\chi_2 + \frac{\tau}{2}(\chi_1 - \chi_1^\dagger)(\alpha + \alpha^\dagger) + \frac{\tau}{2}(\chi_2 + \chi_2^\dagger)(\alpha - \alpha^\dagger)$$

–Electron number conserved modulo 2.

–Begin with electron on dot 1, dot 2 empty.

–Time until electron appears on dot 2 is $\pi/\sqrt{E^2 + \tau^2}$

independent of L

Teleportation

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–“Teleportation”

Teleportation

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independent of L

–"Teleportation" **"Stretching the electron as far as it will go"**

Two spin states

$$\psi_{\uparrow k} = \beta_\uparrow b_k^0 + i\gamma_\uparrow b_{L+1-k}^0 + \ldots \quad , \quad \beta_\sigma = \beta_\sigma^\dagger \ , \ \beta_\sigma^2 = \frac{1}{2}$$

$$\psi_{\downarrow k} = \beta_\downarrow b_k^0 + i\gamma_\downarrow b_{L+1-k}^0 + \ldots \quad , \quad \gamma_\sigma = \gamma_\sigma^\dagger \ , \ \gamma_\sigma^2 = \frac{1}{2}$$

"local operators"

$$a = \frac{\beta_\uparrow + i\beta_\downarrow}{\sqrt{2}} \ , \ a^\dagger = \frac{\beta_\uparrow - i\beta_\downarrow}{\sqrt{2}} \quad , \quad b = \frac{\gamma_\uparrow + i\gamma_\downarrow}{\sqrt{2}} \ , \ b^\dagger = \frac{\gamma_\uparrow - i\gamma_\downarrow}{\sqrt{2}}$$

$$a|0>= 0 \ , \ a^\dagger|0>= |a> \quad ; \quad b|0>= 0 \ , \ b^\dagger|0>= |b>$$

$$\text{states} : \{|a>, |b>\} \ \ (-1)^F = -1 \quad , \quad \{|0>, |ab>\} \ \ (-1)^F = 1$$

Two Bloch spheres:

$$e^{i\phi/2} \cos\frac{\theta}{2}|0> + e^{-i\phi/2} \sin\frac{\theta}{2}|ab> \quad , \quad e^{i\phi/2} \cos\frac{\theta}{2}|a> + e^{-i\phi/2} \sin\frac{\theta}{2}|b>$$

$$SU(2) : \ a^\dagger b^\dagger \ , \ ba \ , \ \frac{a^\dagger a + b^\dagger b}{2} - \frac{1}{2} \qquad SU(2)' : \ ab^\dagger \ , \ ba^\dagger \ , \ \frac{a^\dagger a - b^\dagger b}{2}$$

manipulate phase ϕ with $\psi^\dagger \vec{\sigma} \psi$

$$|\theta,\phi>_e = e^{i\phi/2}\cos\frac{\theta}{2}|0> + e^{-i\phi/2}\sin\frac{\theta}{2}|ab>$$

$$SU(2):\ \ a^\dagger b^\dagger\ ,\ ba\ ,\ \frac{a^\dagger a + b^\dagger b}{2} - \frac{1}{2}$$

$$_e<\theta\phi|\psi^\dagger\psi|\theta\phi>_e = \frac{1}{2}|b_k^0|^2$$

$$_e<\theta'\phi'|\psi^\dagger\sigma^a\psi|\theta\phi>_e = \delta^{a2}\frac{1}{2}\cos\theta|b_k^0|^2 = \delta^{a2}|b_k^0|^2 <\theta\phi|\frac{1}{i}\frac{\partial}{\partial\phi}|\theta\phi>$$

Conclusions:

- Majorana zero modes have interesting non-local effects
- Fermion parity anomaly or spin anomaly
- further applications
- experimental evidence