Boundary Field Theory of Superconducting Devices

D.Giuliano *(Cosenza)*, P. Sodano *(Perugia)*

Perugia, July 2007

Main idea

One-dimensional array of junctions + weak links ⇒ Spinless Luttinger liquid + boundary interaction(s);

Boundary Field Theory ⇒ Phase Diagram, observable quantities;

Phase Diagram ⇒ Better control on the device (more efficient qubit(s)).

Plan of the talk:

1. Mapping onto a 1+1-dimensional boundary field theory: "bulk" phase diagram;

2. The chain with a weak link: boundary interaction and RG flow of the boundary coupling strength;

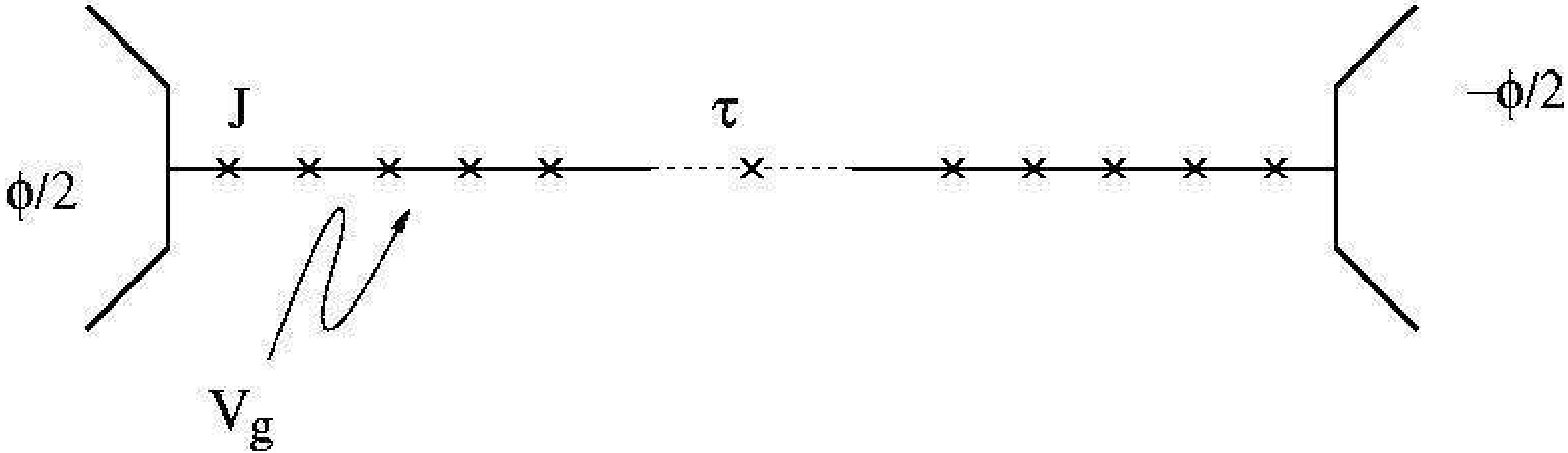
3. Three-chain device: correspondence with the quantum Brownian motion on a triangular lattice;

4. Phase diagram: finite coupling fixed point(s);

5. Possible implementation as (stable) qubit;

6. Conclusions, possible applications, perspectives.

1. Mapping onto a 1+1-dimensional model



$$H_0 = \frac{E_C}{2} \sum_{j=1}^{L} \left[-i \frac{\partial}{\partial \phi_j} - N \right]^2 - J \sum_j \cos\left[\phi_j - \phi_{j+1}\right]$$

Charging energy + Josephson energy

$$(N (\propto V_g) = n + h + \frac{1}{2})$$

$E_C/J >> 1 \Rightarrow$ truncated Hilbert space

$$\left|\{n\}\right\rangle = \prod_{j=1}^{L} \left|n_j\right\rangle$$

$$(n_j = n, n+1)$$

$$P_F = \sum_{\{n\}} \left|\{n\}\right\rangle\left\langle\{n\}\right|$$

Contributions from virtual processes + intergrain capacitance

$$H_2 = \left(E_Z - \frac{3}{16}\frac{J^2}{E_C} \right) \sum_j n_j n_{j+1}$$

Effective, spin-1/2, operators

$$S_j^z = n_j - n - \frac{1}{2}$$

$$S_j^{\pm} = P_F \exp[\pm i\phi_j] P_F$$

$$(H = E_C h, \Delta = E^z - \frac{3}{16}\frac{J^2}{E_C})$$

Anisotropic Heisenberg model in an applied field

$$H_{Eff} = -\frac{J}{2}\sum_j [S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+] + H\sum_j S_j^z + \Delta \sum_j S_j^z S_{j+1}^z$$

Lattice Jordan-Wigner fermions a_j

$$\left(\left\{a_j, a_i^+\right\} = \delta_{ji}\right)$$

$$S_j^z = a_j^+ a_j - \frac{1}{2}$$

$$S_j^+ = a_j^+ \exp\left[i\pi \sum_k^{j-1} a_k^+ a_k\right]$$

$$H_{Eff}^f = -\frac{J}{2}\sum_j [a_j^+ a_{j+1} + a_{j+1}^+ a_j] + H\sum_j a_j^+ a_j$$
$$+ \Delta \sum_j (a_j^+ a_j - \frac{1}{2})(a_{j+1}^+ a_{j+1} - \frac{1}{2})$$

Long-wavelenght expansion

$$\frac{a_j}{\sqrt{2\pi a}} \approx e^{ik_F x_j}\psi_L(x_j) + e^{-ik_F x_j}\psi_R(x_j)$$

Fermionic Hamiltonian

$$H_f \approx -i(v_F + 4\pi\Delta\cos(k_F a))\int\left[\psi_L^+ \frac{\partial}{\partial x_j}\psi_L - \psi_R^+ \frac{\partial}{\partial x_j}\psi_R\right]dx_j$$

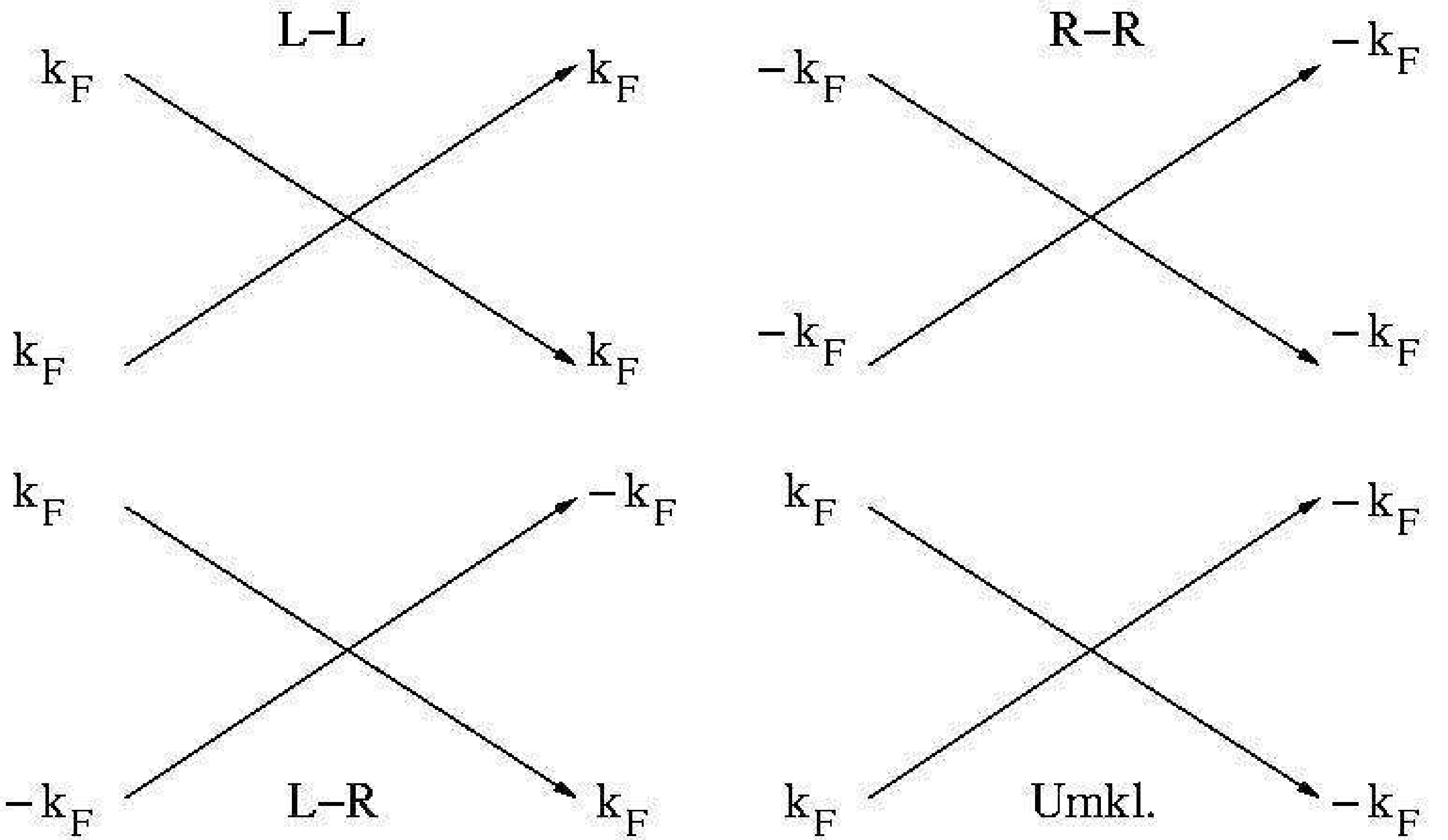
$$+4\pi^2 a\Delta\int\left[(:\psi_L^+\psi_L:)^2 + (:\psi_R^+\psi_R:)^2\right]dx_j$$

$$+16\pi^2 a\Delta sen^2(k_F a)\int(:\psi_L^+\psi_L:)(:\psi_R^+\psi_R:)dx_j$$

Chiral bosonization rules (Fermions→Bosons)

$$:\psi_{L(R)}^+(x)\psi_{L(R)}(x):=\mp\frac{1}{2\pi}\frac{\partial\phi_{L(R)}(x)}{\partial x}$$

Relevant scattering processes



Bosonic Hamiltonian

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$$g = \sqrt{\frac{v_F + g_2 - g_4}{v_F + g_2 + g_4}}$$

$$u = \sqrt{(v_F + g_2)^2 - g_4^{\,2}}$$

$$g_2 = g_4 = 4\pi a\Delta[1 - \cos(2k_F a)]$$

$$G \propto \Delta$$

Basic Fields

$$\Phi = \frac{\phi_R + \phi_L}{\sqrt{2g}}$$

$$\Theta = \sqrt{g}\,\frac{\phi_R - \phi_L}{\sqrt{2}}$$

Bulk phase diagram

Cutoff rescaling: a→a/Λ, Λ>1 ⇒RG flow of the running parameter $\Gamma = G(\Lambda/a)^{2-4g}$

$$\frac{d\Gamma}{d\ln(\Lambda/\Lambda_0)} = (2-4g)\Gamma$$

Umklapp interaction irrelevant for g>1/2. We choose g>1, so, we shall neglect it.

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No real solutions for k_F

$$H - 2E^z - J + \frac{3}{8}\frac{J^2}{E_C} > 0$$

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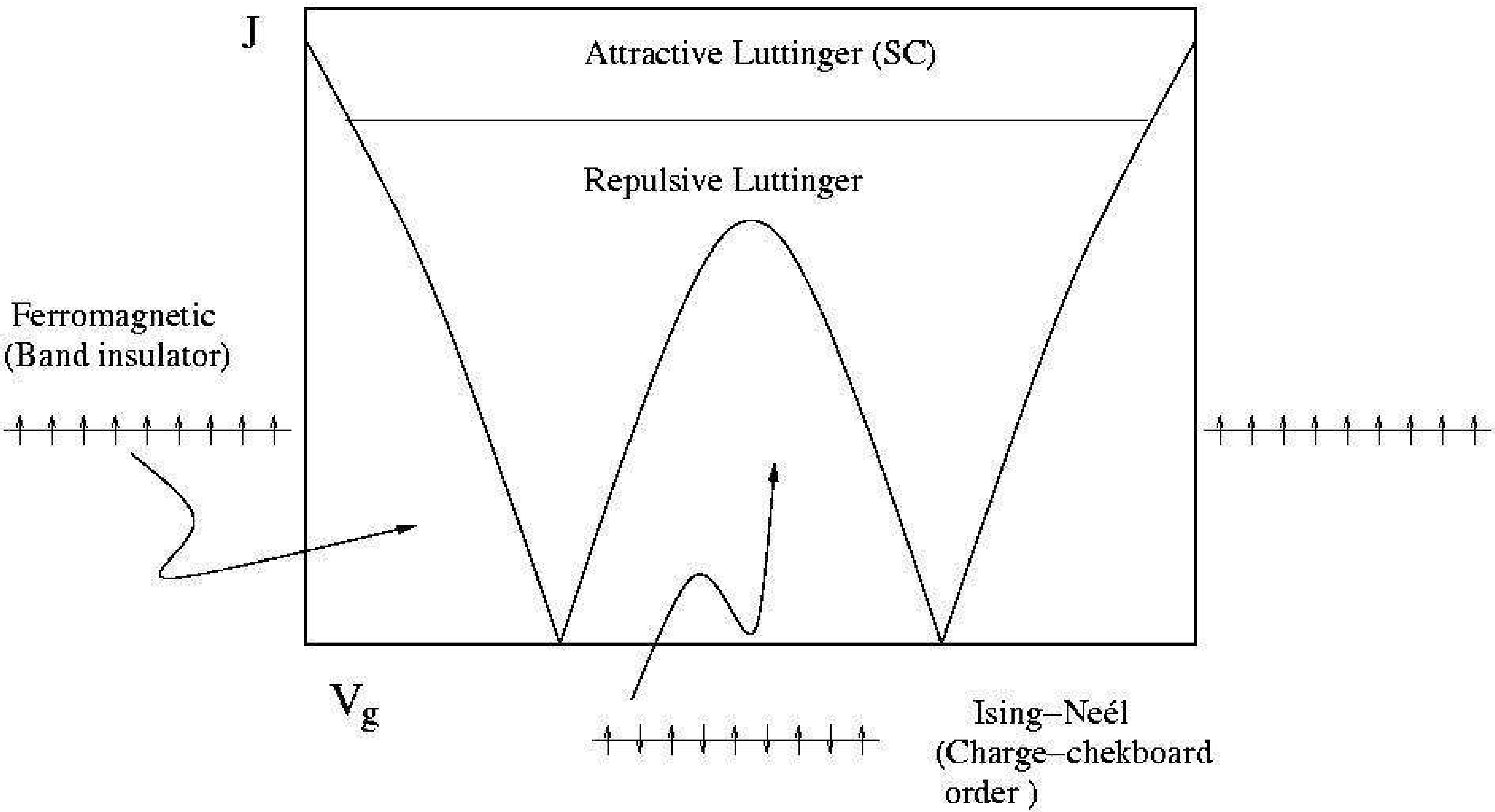
Luttinger liquid phases

$\Delta > 0 \Rightarrow g < 1$: Repulsive Luttinger Liquid

$\Delta < 0 \Rightarrow g > 1$: Attractive Luttinger Liquid

"Bulk" phase diagram of the chain



2. The chain with a weak link

Basic Fields

$$\Phi_{\pm} = \frac{1}{\sqrt{2}}[\Phi_{>} \pm \Phi_{<}]$$

$$\Theta_{\pm} = \frac{1}{\sqrt{2}}[\Theta_{>} \pm \Theta_{<}]$$

Boundary interaction at the weak link

$$H_{\tau} = -\tau[S^{+}_{<,0}S^{-}_{>,0} + S^{+}_{>,0}S^{-}_{<,0}] \rightarrow -E_W :\cos[\Phi\_(0)]:$$

Boundary conditions at the "outer" boundary

$$S^{+}_{L/a,>} = e^{i\varphi/2}$$

$$S^{+}_{-L/a,<} = e^{-i\varphi/2}$$

$$\rightarrow \Phi\_(L) = \varphi$$

RG flow at weak coupling

Neumann Boundary Cs.

$$\frac{\partial \Phi\_(0)}{\partial x} = 0$$

Dimensionless coupling

$$\lambda = \left(\frac{\Lambda}{a}\right)^{1-1/g} E_W$$

g<1⇒perturbative calculation of the Josephson current

$$I[\varphi] = \frac{2e}{c} \lim_{\beta \to 0} \left[-\frac{1}{\beta} \log \left(\frac{Z[E_W]}{Z[E_W = 0]} \right) \right] \propto \sin[\varphi]$$

RG flow at strong coupling

Dirichlet Boundary Cs.

$$\Phi\_(0) = 2\pi n$$

Partition function at sc

$$Z = \frac{1}{\prod_{n>0}\left(1 - e^{-\beta\frac{n\pi v_F}{L}}\right)} \sum_k e^{-\beta\frac{\pi v_F}{L}\left(k - \frac{\varphi}{2\pi}\right)^2}$$

g>1⇒crossover to a sawtooth-like Josephson current

$$I[\varphi] \propto \frac{2eg}{L}(\varphi - [\varphi])$$

Finite-size "inductive" energy

$$E_M[\varphi] = \frac{\pi u g}{L}[n - \frac{\varphi}{2\pi}]^2$$

Degenerate for φ=π+2n π

Leading perturbation at strong coupling (consistent with boundary interaction)

$$H_D = -Y \cos[\Theta(0)]$$

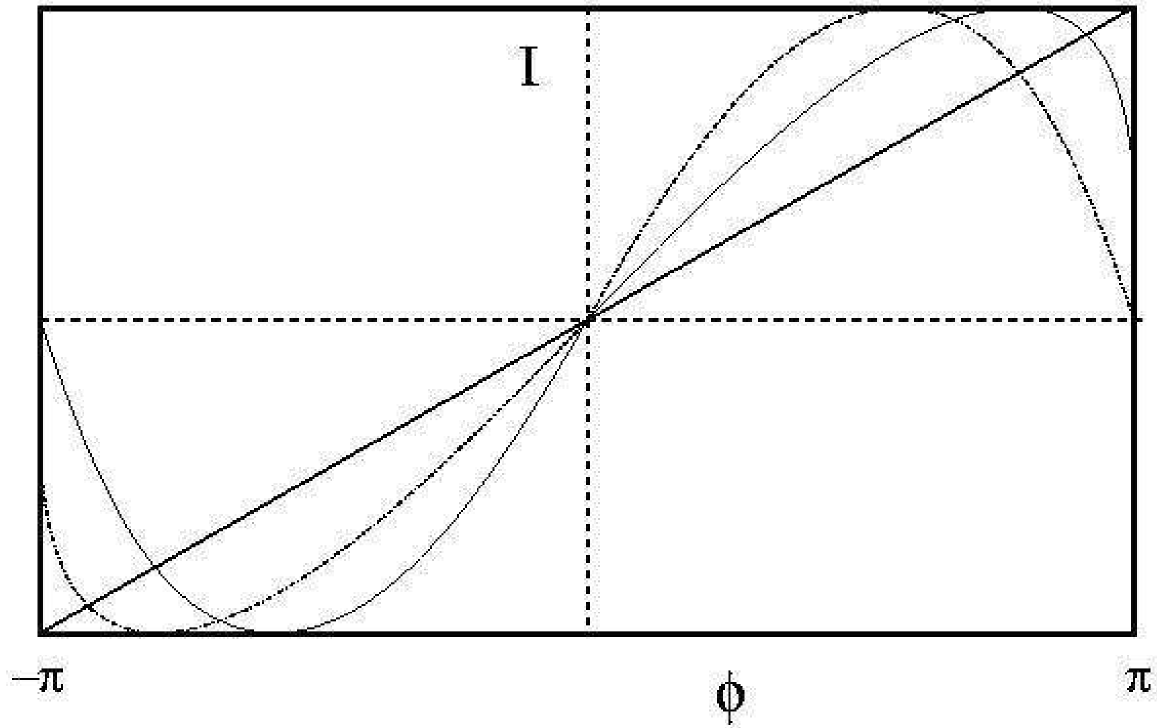
Changes by +/-1 the value of Φ(0): "instanton" trajectories between the minima of H_W+E_M

Dimensionless coupling

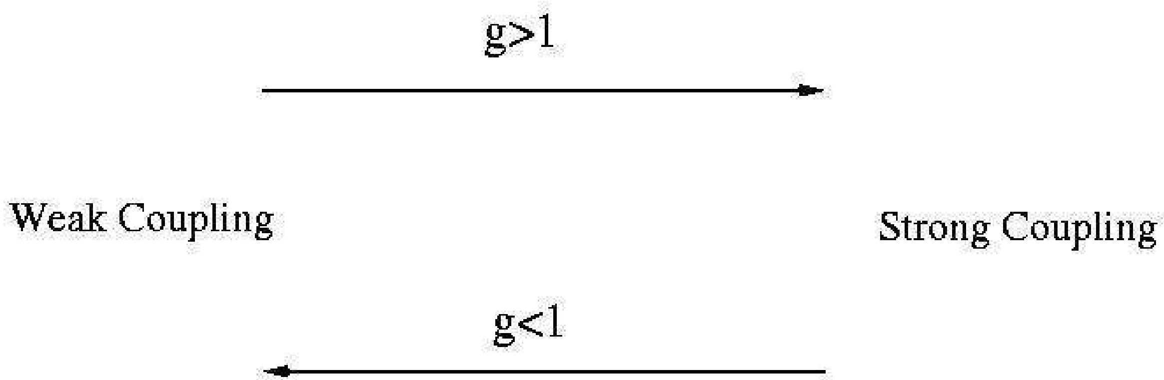
$$y = \left(\frac{\Lambda}{a}\right)^{1-g} Y$$

Instantons: irrelevant for g>1

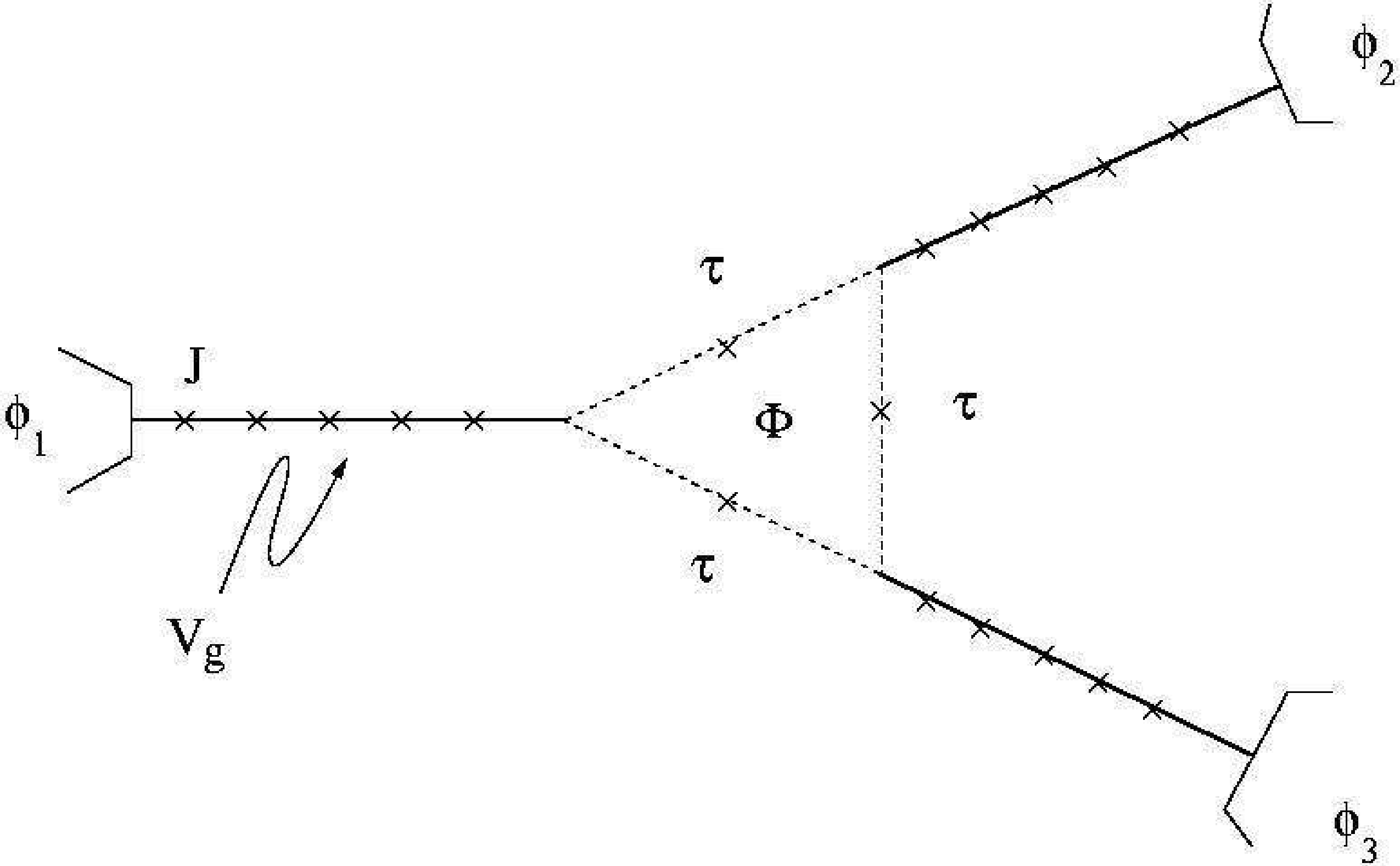
Josephson current



(Minimal) phase diagram



3. Three-chain device



Central region Hamiltonian

$$H_\Delta = \frac{C}{2}\sum_{i=1}^{3}\left[-i\frac{\partial}{\partial\phi_i^{(0)}} - e^* W_g\right]^2 - \frac{\tau}{2}\sum_{i=1}^{3}\left[e^{i(\phi_i^{(0)} - \phi_{i+1}^{(0)} + \varphi/3)} + h.c.\right]$$

Effective (3)-spin Hamiltonian

$$e^* W_g = N + h + \frac{1}{2}$$

$$[S_i^{(0)}]^z = -i\frac{\partial}{\partial\phi_i^{(0)}} - N - \frac{1}{2}$$

$$[S_i^{(0)}]^+ = e^{i\phi_i^{(0)}}$$

$$H_\Delta = -h\sum_{i=1}^{3}[S_i^{(0)}]^z - \frac{\tau}{2}\sum_{i=1}^{3}\left\{[S_i^{(0)}]^+[S_{i+1}^{(0)}]^- e^{i\varphi/3} + h.c.\right\}$$

Low-energy eigenstates (h>τ)

$$|\uparrow\uparrow\uparrow\rangle \qquad \varepsilon_0 = -\tfrac{3}{2}h$$

$$\frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle] \qquad \varepsilon_{11} = -\tfrac{1}{2}h - \frac{\tau}{2}\cos(\frac{\varphi}{3})$$

$$\frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow\rangle - e^{-i\frac{\pi}{3}}|\uparrow\downarrow\uparrow\rangle - e^{i\frac{\pi}{3}}|\downarrow\uparrow\uparrow\rangle] \qquad \varepsilon_{12} = -\tfrac{1}{2}h - \frac{\tau}{2}\cos(\frac{\varphi-\pi}{3})$$

$$\frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow\rangle - e^{i\frac{\pi}{3}}|\uparrow\downarrow\uparrow\rangle - e^{-i\frac{\pi}{3}}|\downarrow\uparrow\uparrow\rangle] \qquad \varepsilon_{13} = -\tfrac{1}{2}h - \frac{\tau}{2}\cos(\frac{\varphi+\pi}{3})$$

Only these states will be kept in the effective theory

Charge tunneling with the three chain endpoints

$$H_T = -\lambda \sum_{i=1}^{3} \cos[\phi_i^{(0)} - \phi_i^{(1)}]$$

"Weak tunneling" limit: λ<<h,J ⇒Schrieffer-Wolff transformation ⇒Boundary interaction term

$$H_B = -\Omega \sum_{i=1}^{3} [e^{i(\phi_i^{(1)} - \phi_{i+1}^{(1)})} e^{i\chi} + h.c.]$$

$$\Omega \approx \frac{\lambda^2 \tau}{48 h^2} \sqrt{[\cos^2(\frac{\varphi}{3}) + 9 \sin^2(\frac{\varphi}{3})}$$

$$\chi = \arctan[3 \tan(\frac{\Phi}{3})]$$

Continuum limit of the effective Hamiltonian

$$H = H_{Bulk} + H_{Bou}$$

$$H_{Bulk} = \frac{g}{4\pi} \sum_{i=1}^{3} \int_0^L \left[u \left(\frac{\partial \Phi_i}{\partial x} \right)^2 + \frac{u}{g} \left(\frac{\partial \Theta_i}{\partial x} \right)^2 \right]$$

$$\Phi_i(L) = \phi_i$$

$$H_{Bou} = -2E \sum_{i=1}^{3} \cos[\Phi_i(0) - \Phi_{i+1}(0) + \chi]$$

“Normal” fields

$$X(x) = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} \Phi_i(x)$$

$$\psi_1(L) = \frac{\phi_1 - \phi_2}{\sqrt{2}}$$

$$\psi_1(x) = \frac{1}{\sqrt{2}} [\Phi_1(x) - \Phi_2(x)]$$

$$\psi_2(L) = \frac{\phi_1 + \phi_2 - 2\phi_3}{\sqrt{6}}$$

$$\psi_2(x) = \frac{1}{\sqrt{6}} [\Phi_1(x) + \Phi_2(x) - 2\Phi_3(x)]$$

$$H_{Bou} = -2E \sum_{i=1}^{3} \exp[i(\vec{\alpha} \bullet \vec{\psi}(0) + \chi)] + h.c.$$

Quantum Brownian motion on a frustrated triangular lattice

(Affleck, Oshikawa, Saleur; Kane, Yi)

4. Phase diagram

RG flow at weak coupling

Neumann boundary conditions

$$\frac{\partial \psi_j(0)}{\partial x} = 0$$

Dimensionless coupling

$$\lambda = \left(\frac{\Lambda}{a}\right)^{1-1/g} E$$

Second-order RG equations

$$\frac{d\lambda}{d\ln(\Lambda/\Lambda_0)} = \left(1-\frac{1}{g}\right)\lambda - 2\lambda^2$$

$$\chi = \pi/3$$

g<1 ⇒ perturbative calculation of the Joseph. current

$$I_i = -\frac{e^*}{\beta}\lim_{\beta\to\infty}\frac{\partial \ln Z[\{\phi_i\}]}{\partial\phi_i} = e^* 2\overline{E}\{\sin[\phi_i - \phi_{i+1} + \chi] - \sin[\phi_i - \phi_{i-1} + \chi]\}$$

Renormalized coupling

$$\overline{E} = \left(\frac{\Lambda_0}{\Lambda}\right)^{\frac{1}{g}} E$$

(J.E.Mooij et al., Science 285 (1036), but with renormalized coupling)

RG flow at strong coupling

Dirichlet Boundary Cs.

$$(\psi_1(0),\psi_2(0)) = (2\pi m_1, \frac{2}{\sqrt{3}}[2\pi m_2 + \pi m_1])$$

Sublattice A

$$(\psi_1(0),\psi_2(0)) = (2\pi m_1 - \frac{2\pi}{3}, \frac{2}{\sqrt{3}}[2\pi m_2 + \pi m_1 - \pi])$$

Sublattice B

Partition function

$$Z = Z_{Osc} Z_{0-\mathrm{mode}}$$

$$Z_{osc} = \prod_{n=1}^{\infty} \left[\frac{1}{1 - e^{-\beta \frac{\pi u n}{L}}} \right]^2 \left[\frac{1}{1 - e^{-\beta \frac{\pi u (n-1/2)}{L}}} \right]$$

$$Z_{0-\mathrm{mode}} = \sum_{m_1,m_2}\left\{-\frac{\beta g u}{4\pi L}\left[(2\pi m_1+\alpha_1)^2+\frac{2}{3}(2\pi m_2+\pi m_1+\alpha_2)^2\right]\right\}+$$
$$\sum_{m_1,m_2}\left\{-\frac{\beta g u}{4\pi L}\left[(2\pi m_1-\frac{2\pi}{3}+\alpha_1)^2+\frac{2}{3}(2\pi m_2+\pi m_1-\pi+\alpha_2)^2\right]\right\}$$

$$\alpha_1 = \phi_2 - \phi_1$$

$$\alpha_2 = \frac{1}{\sqrt{3}}[2\phi_3 - \phi_1 - \phi_2]$$

Josephson current

$$I_i = -\lim_{\beta\to\infty}\frac{e^*}{\beta}\frac{\partial \ln Z_{0-\mathrm{mode}}}{\partial \phi_i}$$

Sawtooth-like behavior

Leading perturbation at strong coupling (consistent with boundary interaction)

$$H_D = -Y\sum_{i=1}^{3}\left\{\tau^{+}\exp[i\frac{2}{3}\vec{\alpha}_j\cdot\vec{\Theta}(0)] + \tau^{-}\exp[-i\frac{2}{3}\vec{\alpha}_j\cdot\vec{\Theta}(0)]\right\}$$

$$+\frac{Y^2}{2\pi u}\tau^{z}\sum_{i=1}^{3}\frac{\partial\vec{\Theta}(0)}{\partial t}\cdot\vec{\alpha}_i$$

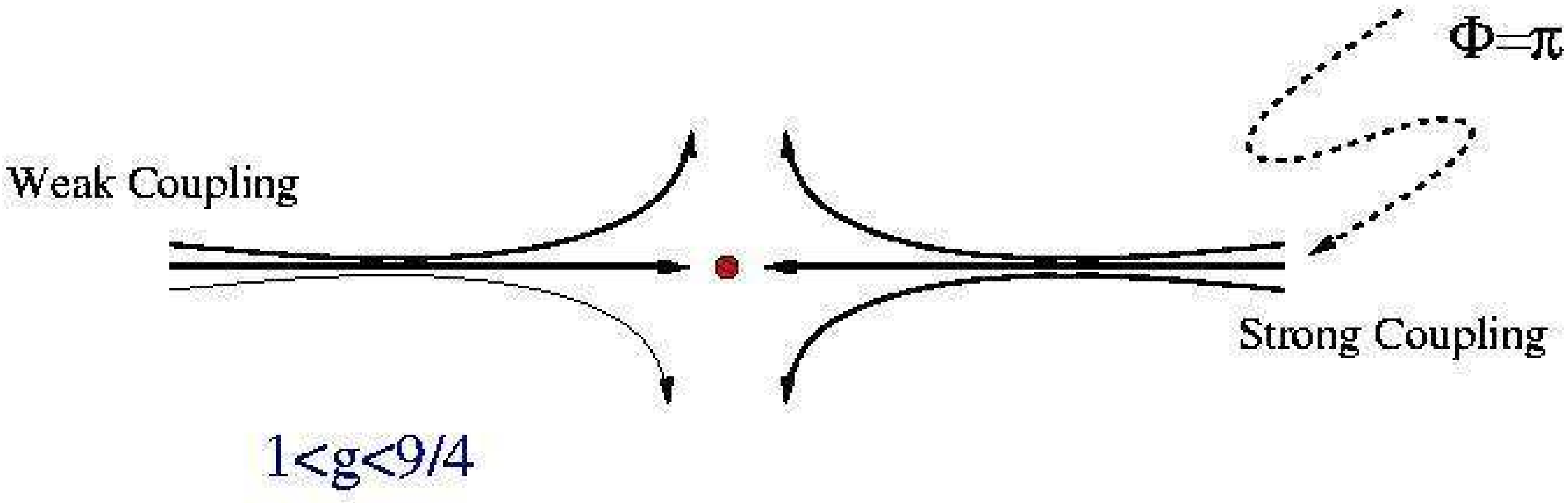
Dimensionless coupling

$$y = \left(\frac{\Lambda}{a}\right)^{1-4g/9} Y$$

RG equation

$$\frac{dy}{d\ln(\Lambda/\Lambda_0)} = \left(1-\frac{4}{9}g\right)y - y^3$$

(Minimal) phase diagram



Stable finite-coupling fixed point (at χ=π/3)

(Kane-Yi fixed point)

5. Possible implementation

Phases φ_i =control parameneters (selection of a desired pair of minima)

$$\alpha_1 = -\frac{2\pi}{3}; \alpha_2 = 0$$

Degeneracy between (0,0) on sublattice A and (1,0) on sublattice B

$$I_1 = -I_2 = \pm\frac{gu}{6\pi L}; I_3 = 0$$

Degenerate states

$$\alpha_1 = -\frac{2\pi}{3} + h$$

h=control parameter (flipping between states

Effective mapping on a ferromagnetic Kondo Hamiltonian + exact fermionization at g=9/8 ⇒exact calculation of Josephson current

$$\Theta(x,t) = \sqrt{\frac{g}{2}}[\theta_R(x-ut)+\theta_L(x+ut)]$$

$$\phi(x-ut) = \theta_R(x-ut)(0<x<L);$$
$$= \theta_L(|x|-ut)(-L<x<0).$$

(Re)fermionization

$$\chi(x-ut) =: e^{-i\phi(x-ut)}:$$

$$\tau^z = d^+d - \frac{1}{2}$$

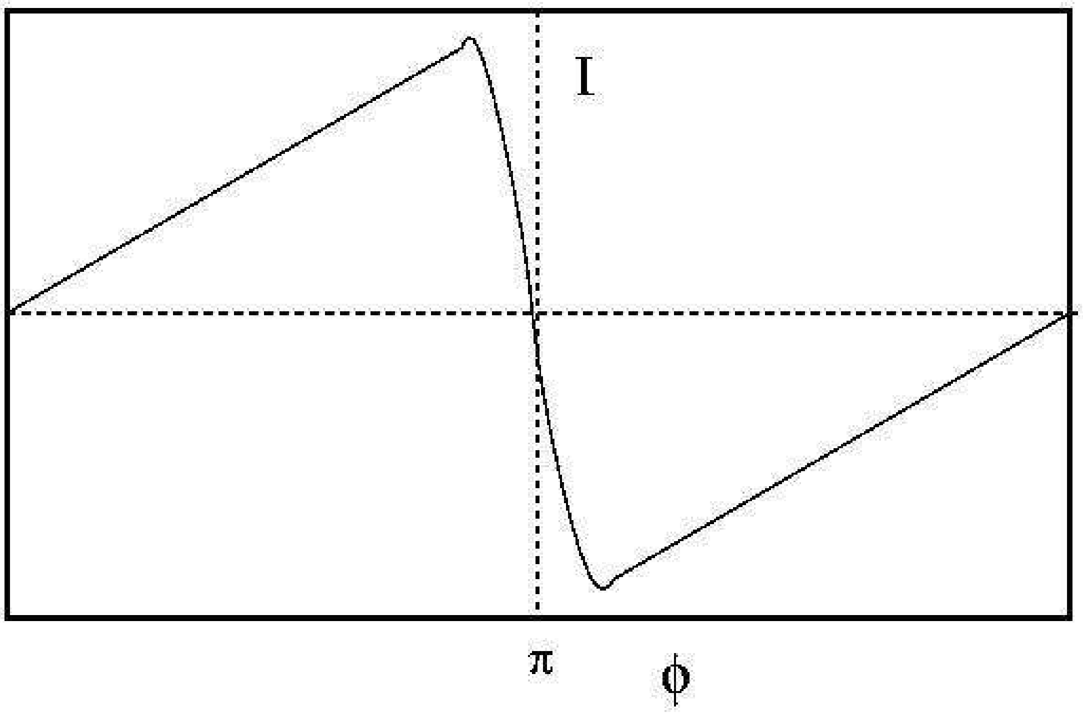
$$\tau^+ = d^+$$

$$H_f = -iu\int_{-L}^{L}\chi^+(x)\frac{d\chi(x)}{dx}dx - Y[\chi^+(0)d + d^+\chi(0)]$$

Josephson current

$$I = \frac{9}{8}\frac{uh}{\pi L} + \frac{\partial \varepsilon[h]}{\partial h}$$

$$\tan\left[\frac{\varepsilon L}{u} + \frac{3h}{4}\right] = \frac{u}{Y^2}\left[-\varepsilon + \frac{3hu}{4L} + \frac{\pi u}{2L}\right]$$



Same as with the one-weak link chain, but now instanton are a relevant perturbation for g>1

6. Conclusions and further perspectives

a. BCFT approach: possibility of realizing Josephson devices with nontrivial phase diagrams;

b. Finite-coupling fixed-point based qubit: robust quantum coherence of the degenerate states (remarkable improvement with respect, e.g., rf-SQUIDS);

c. Effects of noise (D.G., P. Sodano, work in preparation);

d. More realistic models (interaction between chains and researvoirs: Affleck-Zagoskin-Caux model);

e. Efficient way to apply the phases φ_i (d-wave sc?).

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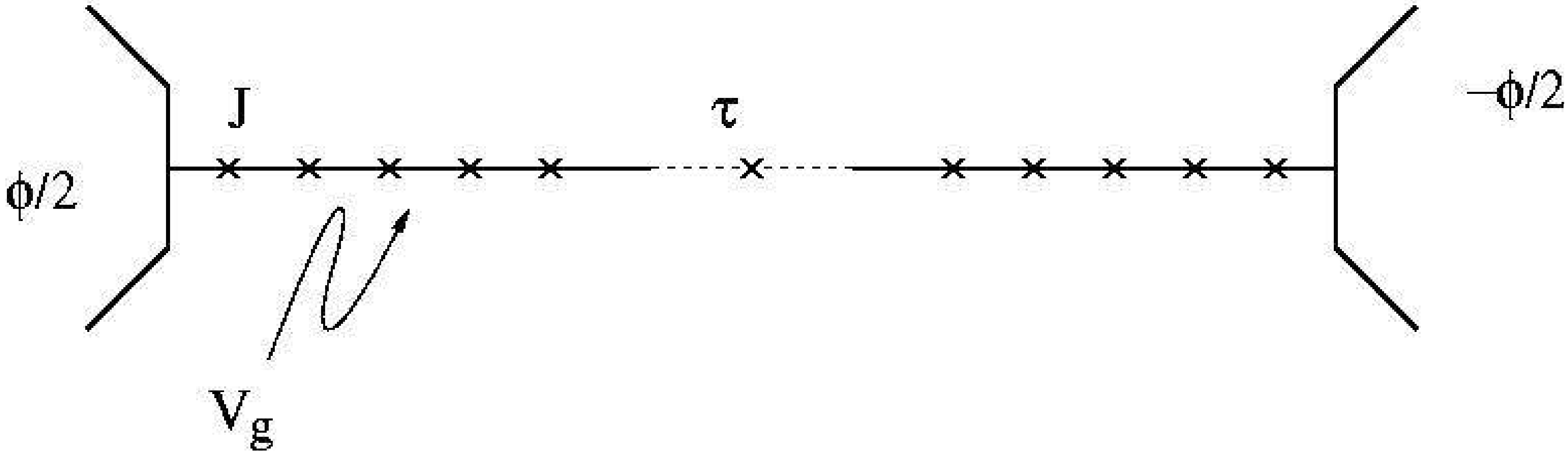

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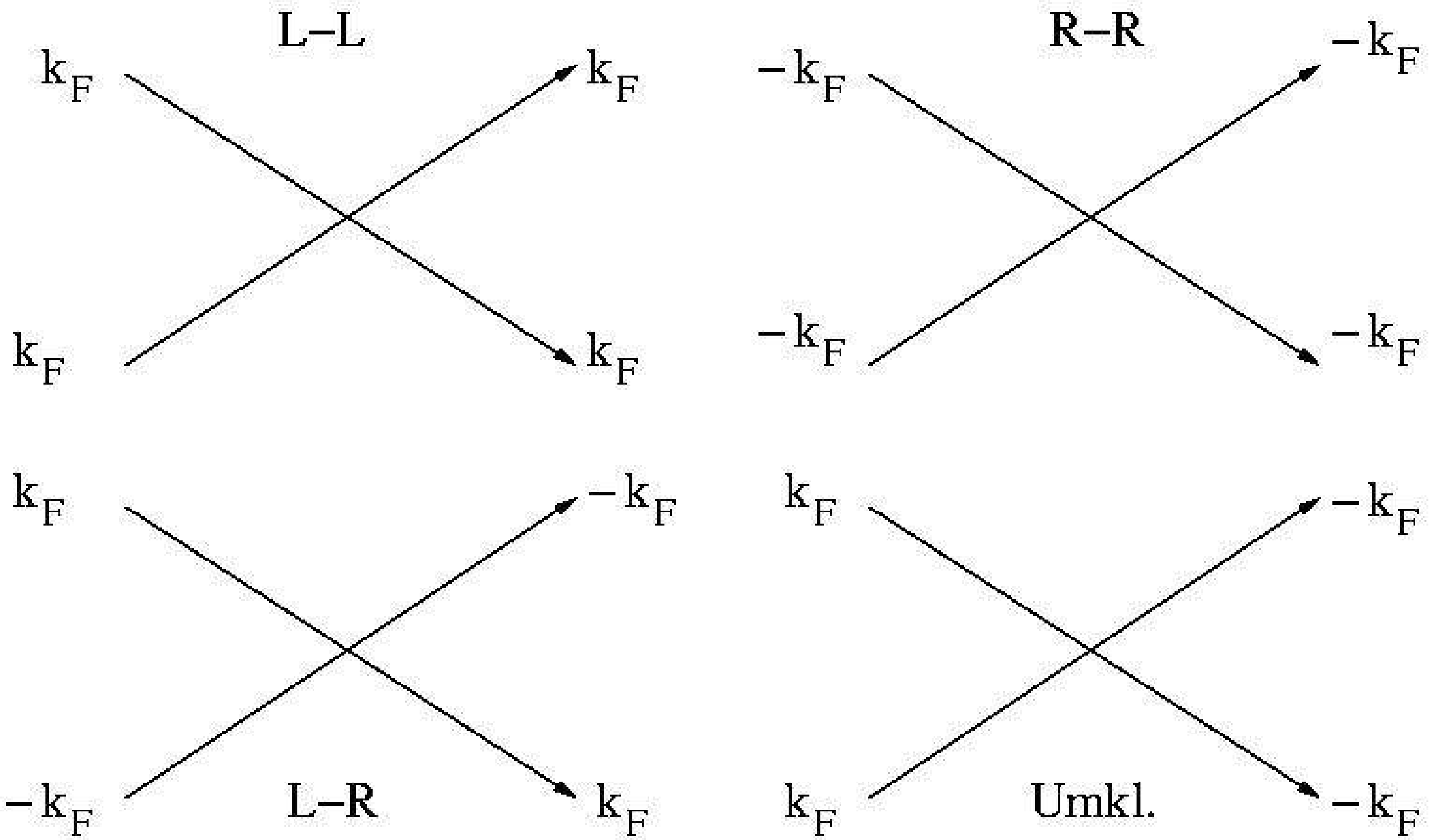

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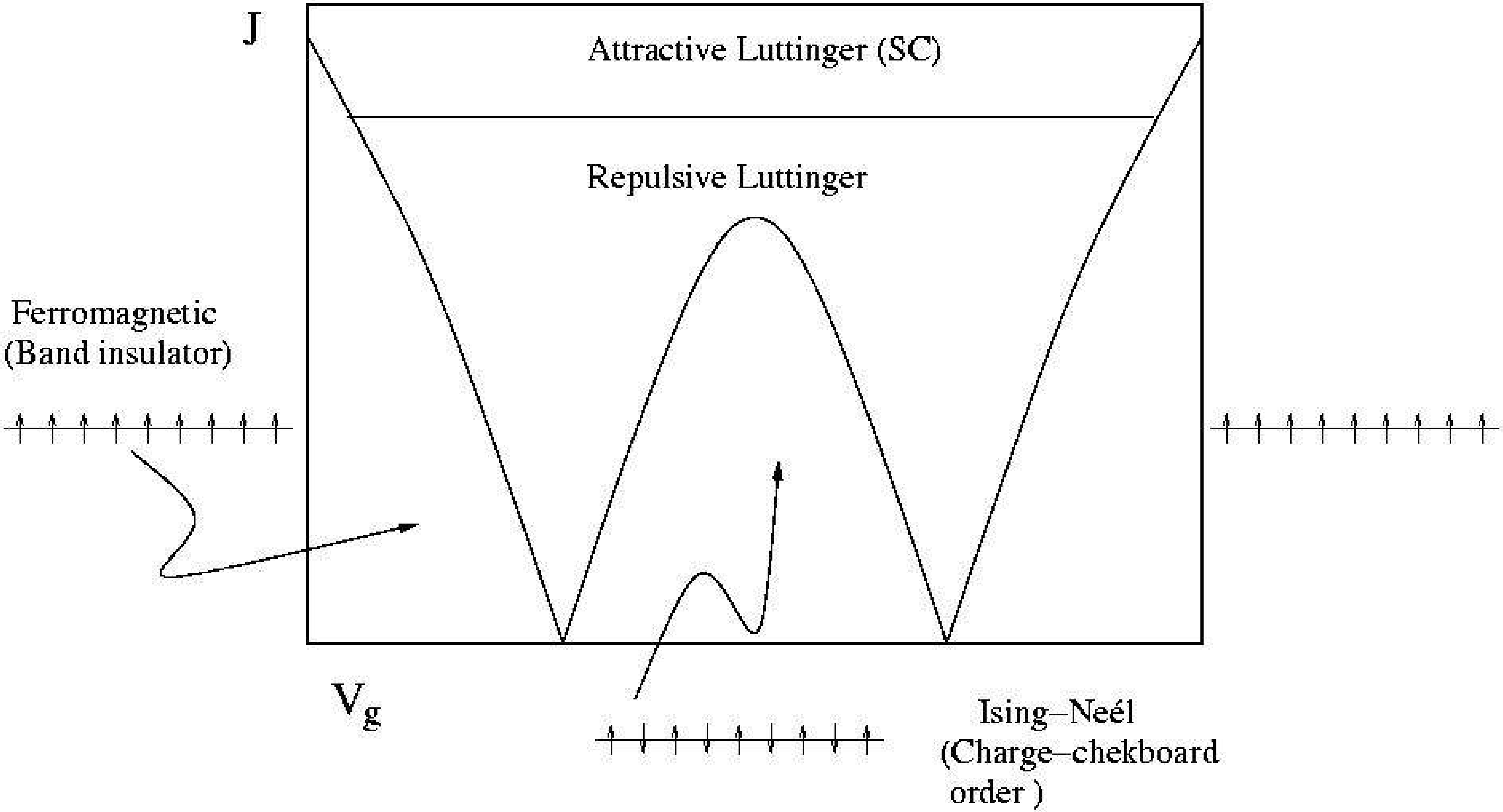

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$$\Theta_{\pm} = \frac{1}{\sqrt{2}}[\Theta_{>} \pm \Theta_{<}]$$

Boundary interaction at the weak link

$$H_{\tau} = -\tau[S^{+}_{<,0}S^{-}_{>,0} + S^{+}_{>,0}S^{-}_{<,0}] \to -E_{W} : \cos[\Phi\_(0)] :$$

Boundary conditions at the "outer" boundary

$$S^{+}_{L/a,>} = e^{i\varphi/2}$$

$$S^{+}_{-L/a,<} = e^{-i\varphi/2}$$

$$\to \Phi\_(L) = \varphi$$

RG flow at weak coupling

Neumann Boundary Cs.

$$\frac{\partial \Phi\_(0)}{\partial x} = 0$$

Dimensionless coupling

$$\lambda = \left(\frac{\Lambda}{a}\right)^{1-1/g} E_W$$

g<1⇒perturbative calculation of the Josephson current

$$I[\varphi] = \frac{2e}{c} \lim_{\beta \to 0} \left[-\frac{1}{\beta} \log\left(\frac{Z[E_W]}{Z[E_W = 0]} \right) \right] \propto \sin[\varphi]$$

RG flow at strong coupling

Dirichlet Boundary Cs.

$$\Phi\_(0) = 2\pi n$$

Partition function at sc

$$Z = \frac{1}{\prod_{n>0}\left(1 - e^{-\beta \frac{n\pi v_F}{L}}\right)} \sum_k e^{-\beta \frac{\pi v_F}{L}\left(k - \frac{\varphi}{2\pi}\right)^2}$$

g>1⇒crossover to a sawtooth-like Josephson current

$$I[\varphi] \propto \frac{2eg}{L}(\varphi - [\varphi])$$

Finite-size "inductive" energy

$$E_M[\varphi] = \frac{\pi u g}{L}[n - \frac{\varphi}{2\pi}]^2$$

Degenerate for φ=π+2n π

Leading perturbation at strong coupling (consistent with boundary interaction)

$$H_D = -Y \cos[\Theta(0)]$$

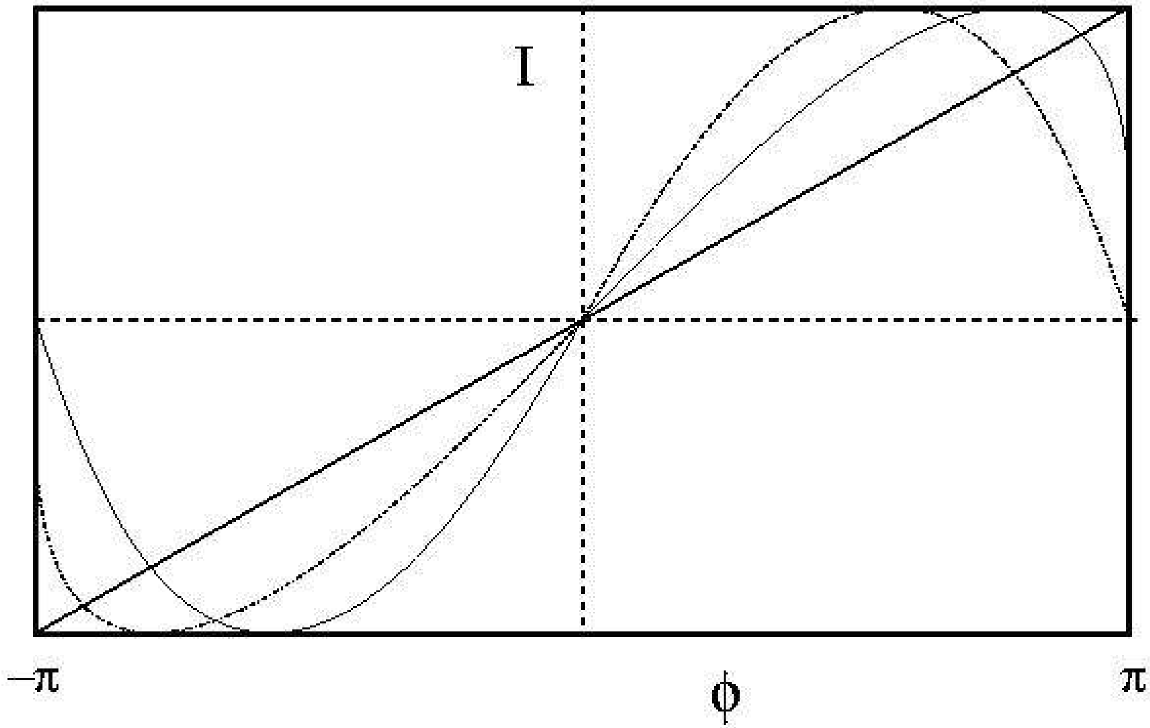
Changes by +/-1 the value of Φ(0): "instanton" trajectories between the minima of H_W+E_M

Dimensionless coupling

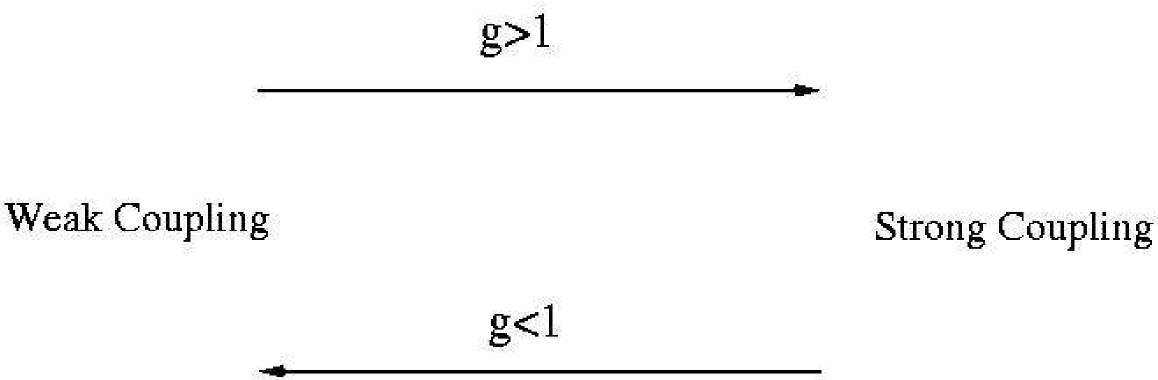
$$y = \left(\frac{\Lambda}{a}\right)^{1-g} Y$$

Instantons: irrelevant for g>1

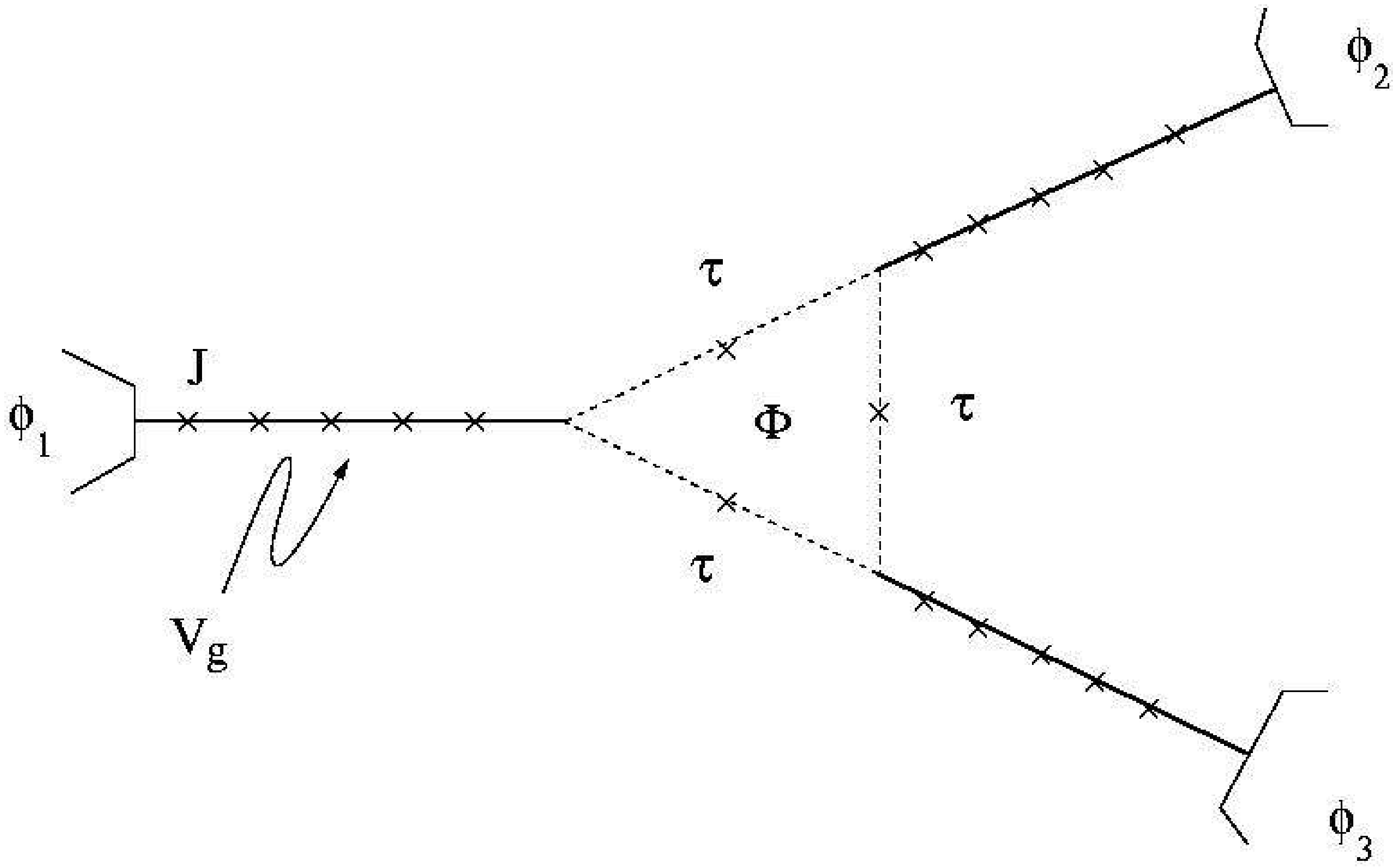
Josephson current



(Minimal) phase diagram



3. Three-chain device



Central region Hamiltonian

$$H_\Delta = \frac{C}{2}\sum_{i=1}^{3}\left[-i\frac{\partial}{\partial\phi_i^{(0)}} - e^* W_g\right]^2 - \frac{\tau}{2}\sum_{i=1}^{3}\left[e^{i(\phi_i^{(0)}-\phi_{i+1}^{(0)}+\varphi/3)} + h.c.\right]$$

Effective (3)-spin Hamiltonian

$$e^* W_g = N + h + \frac{1}{2}$$

$$[S_i^{(0)}]^z = -i\frac{\partial}{\partial\phi_i^{(0)}} - N - \frac{1}{2}$$

$$[S_i^{(0)}]^+ = e^{i\phi_i^{(0)}}$$

$$H_\Delta = -h\sum_{i=1}^{3}[S_i^{(0)}]^z - \frac{\tau}{2}\sum_{i=1}^{3}\left\{[S_i^{(0)}]^+[S_{i+1}^{(0)}]^- e^{i\varphi/3} + h.c.\right\}$$

Low-energy eigenstates (h>τ)

$$|\uparrow\uparrow\uparrow\rangle \qquad \varepsilon_0 = -\tfrac{3}{2}h$$

$$\frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle] \qquad \varepsilon_{11} = -\tfrac{1}{2}h - \frac{\tau}{2}\cos(\frac{\Phi}{3})$$

$$\frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow\rangle - e^{-i\frac{\pi}{3}}|\uparrow\downarrow\uparrow\rangle - e^{i\frac{\pi}{3}}|\downarrow\uparrow\uparrow\rangle] \qquad \varepsilon_{12} = -\tfrac{1}{2}h - \frac{\tau}{2}\cos(\frac{\Phi-\pi}{3})$$

$$\frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow\rangle - e^{i\frac{\pi}{3}}|\uparrow\downarrow\uparrow\rangle - e^{-i\frac{\pi}{3}}|\downarrow\uparrow\uparrow\rangle] \qquad \varepsilon_{13} = -\tfrac{1}{2}h - \frac{\tau}{2}\cos(\frac{\Phi+\pi}{3})$$

Only these states will be kept in the effective theory

Charge tunneling with the three chain endpoints

$$H_T = -\lambda \sum_{i=1}^{3} \cos[\phi_i^{(0)} - \phi_i^{(1)}]$$

"Weak tunneling" limit: λ<<h,J ⇒ Schrieffer-Wolff transformation ⇒ Boundary interaction term

$$H_B = -\Omega \sum_{i=1}^{3} [e^{i(\phi_i^{(1)} - \phi_{i+1}^{(1)})} e^{i\chi} + h.c.]$$

$$\Omega \approx \frac{\lambda^2 \tau}{48 h^2} \sqrt{[\cos^2(\frac{\Phi}{3}) + 9 \sin^2(\frac{\Phi}{3})}$$

$$\chi = \arctan[3 \tan(\frac{\Phi}{3})]$$

Continuum limit of the effective Hamiltonian

$$H = H_{Bulk} + H_{Bou}$$

$$H_{Bulk} = \frac{g}{4\pi}\sum_{i=1}^{3}\int_{0}^{L}\left[u\left(\frac{\partial\Phi_i}{\partial x}\right)^2 + \frac{u}{g}\left(\frac{\partial\Theta_i}{\partial x}\right)^2\right]$$

$$\Phi_i(L) = \phi_i$$

$$H_{Bou} = -2E\sum_{i=1}^{3}\cos[\Phi_i(0) - \Phi_{i+1}(0) + \chi]$$

"Normal" fields

$$X(x) = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} \Phi_i(x)$$

$$\psi_1(x) = \frac{1}{\sqrt{2}} [\Phi_1(x) - \Phi_2(x)]$$

$$\psi_2(x) = \frac{1}{\sqrt{6}} [\Phi_1(x) + \Phi_2(x) - 2\Phi_3(x)]$$

$$\psi_1(L) = \frac{\phi_1 - \phi_2}{\sqrt{2}}$$

$$\psi_2(L) = \frac{\phi_1 + \phi_2 - 2\phi_3}{\sqrt{6}}$$

$$H_{Bou} = -2E \sum_{i=1}^{3} \exp[i(\vec{\alpha} \bullet \vec{\psi}(0) + \chi)] + h.c.$$

Quantum Brownian motion on a frustrated triangular lattice

(Affleck, Oshikawa, Saleur; Kane, Yi)

4. Phase diagram

RG flow at weak coupling

Neumann boundary conditions

$$\frac{\partial \psi_j(0)}{\partial x} = 0$$

Dimensionless coupling

$$\lambda = \left(\frac{\Lambda}{a}\right)^{1-1/g} E$$

Second-order RG equations

$$\frac{d\lambda}{d\ln(\Lambda/\Lambda_0)} = \left(1 - \frac{1}{g}\right)\lambda - 2\lambda^2$$

$$\chi = \pi / 3$$

g<1 ⇒perturbative calculation of the Joseph. current

$$I_i = -\frac{e^*}{\beta} \lim_{\beta\to\infty} \frac{\partial \ln Z[\{\phi_i\}]}{\partial \phi_i} = e^* 2\overline{E}\{\sin[\phi_i - \phi_{i+1} + \chi] - \sin[\phi_i - \phi_{i-1} + \chi]\}$$

Renormalized coupling

$$\overline{E} = \left(\frac{\Lambda_0}{\Lambda}\right)^{\frac{1}{g}} E$$

(J.E.Mooij et al., Science 285 (1036), but with renormalized coupling)

RG flow at strong coupling

Dirichlet Boundary Cs.

$$(\psi_1(0),\psi_2(0)) = (2\pi m_1, \frac{2}{\sqrt{3}}[2\pi m_2 + \pi m_1])$$

Sublattice A

$$(\psi_1(0),\psi_2(0)) = (2\pi m_1 - \frac{2\pi}{3}, \frac{2}{\sqrt{3}}[2\pi m_2 + \pi m_1 - \pi])$$

Sublattice B

Partition function

$$Z = Z_{Osc} Z_{0-\mathrm{mode}}$$

$$Z_{osc} = \prod_{n=1}^{\infty} \left[\frac{1}{1 - e^{-\beta \frac{\pi u n}{L}}} \right]^2 \left[\frac{1}{1 - e^{-\beta \frac{\pi u (n - 1/2)}{L}}} \right]$$

$$Z_{0-\text{mode}} = \sum_{m_1, m_2} \left\{ -\frac{\beta g u}{4\pi L} \left[(2\pi m_1 + \alpha_1)^2 + \frac{2}{3}(2\pi m_2 + \pi m_1 + \alpha_2)^2 \right] \right\} +$$

$$\sum_{m_1, m_2} \left\{ -\frac{\beta g u}{4\pi L} \left[(2\pi m_1 - \frac{2\pi}{3} + \alpha_1)^2 + \frac{2}{3}(2\pi m_2 + \pi m_1 - \pi + \alpha_2)^2 \right] \right\}$$

$$\alpha_1 = \phi_2 - \phi_1$$

$$\alpha_2 = \frac{1}{\sqrt{3}}[2\phi_3 - \phi_1 - \phi_2]$$

Josephson current

$$I_i = -\lim_{\beta \to \infty} \frac{e^*}{\beta} \frac{\partial \ln Z_{0-\text{mode}}}{\partial \phi_i}$$

Sawtooth-like behavior

Leading perturbation at strong coupling (consistent with boundary interaction)

$$H_D = -Y\sum_{i=1}^{3}\left\{\tau^{+}\exp[i\frac{2}{3}\vec{\alpha}_j\cdot\vec{\Theta}(0)] + \tau^{-}\exp[-i\frac{2}{3}\vec{\alpha}_j\cdot\vec{\Theta}(0)]\right\}$$

$$+\frac{Y^2}{2\pi u}\tau^{z}\sum_{i=1}^{3}\frac{\partial\vec{\Theta}(0)}{\partial t}\cdot\vec{\alpha}_i$$

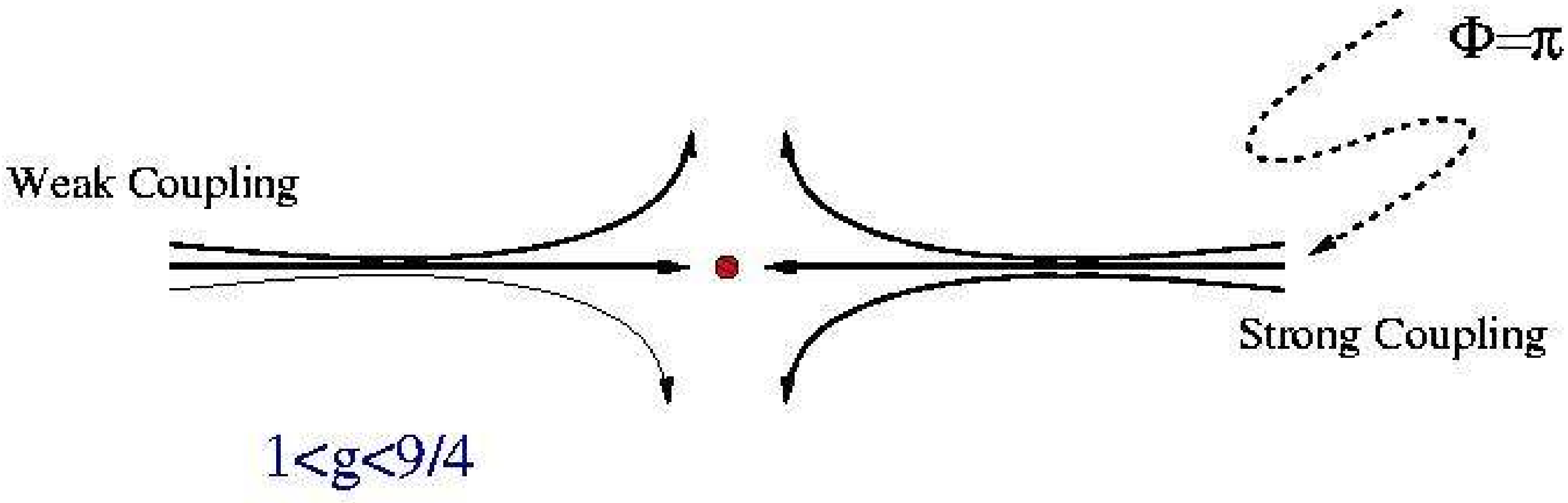
Dimensionless coupling

$$y = \left(\frac{\Lambda}{a}\right)^{1-4g/9} Y$$

RG equation

$$\frac{dy}{d\ln(\Lambda/\Lambda_0)} = \left(1-\frac{4}{9}g\right)y - y^3$$

(Minimal) phase diagram



Stable finite-coupling fixed point (at χ=π/3)

(Kane-Yi fixed point)

5. Possible implementation

Phases φ_i =control parameneters (selection of a desired pair of minima)

$$\alpha_1 = -\frac{2\pi}{3}; \alpha_2 = 0$$

Degeneracy between (0,0) on sublattice A and (1,0) on sublattice B

$$I_1 = -I_2 = \pm \frac{gu}{6\pi L}; I_3 = 0$$

Degenerate states

$$\alpha_1 = -\frac{2\pi}{3} + h$$

h=control parameter (flipping between states

Effective mapping on a ferromagnetic Kondo Hamiltonian + exact fermionization at g=9/8 ⇒exact calculation of Josephson current

$$\Theta(x,t)=\sqrt{\frac{g}{2}}[\theta_R(x-ut)+\theta_L(x+ut)]$$

$$\phi(x-ut)=\theta_R(x-ut)(0<x<L);$$
$$=\theta_L(|x|-ut)(-L<x<0).$$

(Re)fermionization

$$\chi(x-ut)=:e^{-i\phi(x-ut)}:$$

$$\tau^z=d^+d-\frac{1}{2}$$

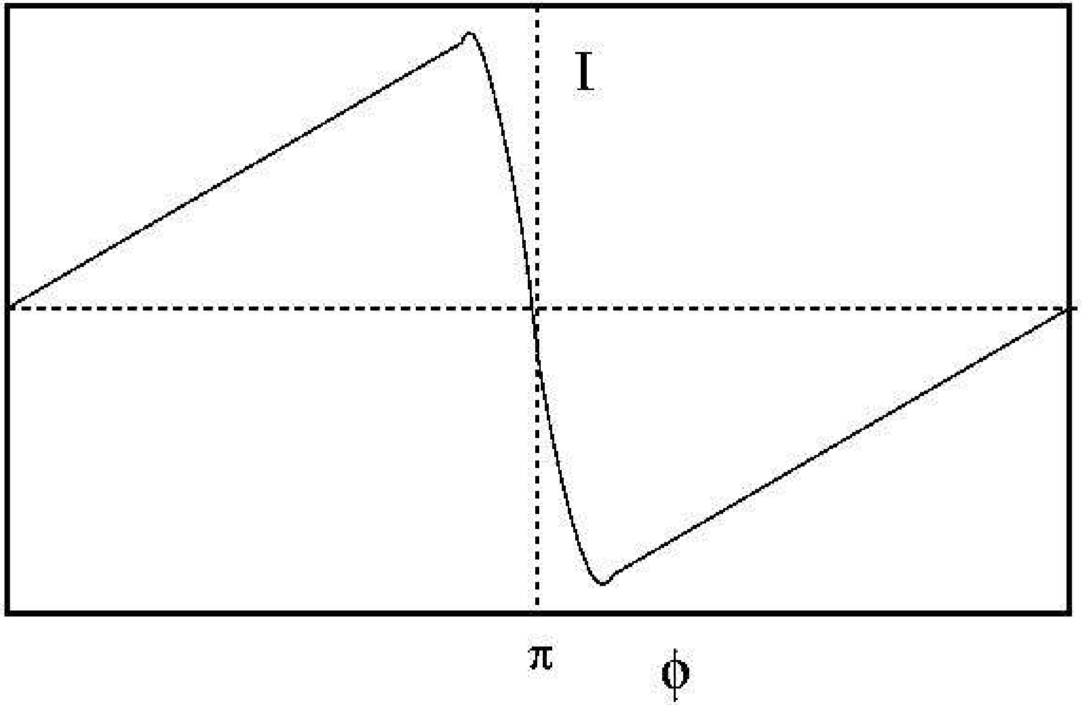
$$\tau^+=d^+$$

$$H_f=-iu\int_{-L}^{L}\chi^+(x)\frac{d\chi(x)}{dx}dx-Y[\chi^+(0)d+d^+\chi(0)]$$

Josephson current

$$I = \frac{9}{8}\frac{uh}{\pi L} + \frac{\partial \varepsilon[h]}{\partial h}$$

$$\tan\left[\frac{\varepsilon L}{u} + \frac{3h}{4}\right] = \frac{u}{Y^2}\left[-\varepsilon + \frac{3hu}{4L} + \frac{\pi u}{2L}\right]$$



Same as with the one-weak link chain, but now instanton are a relevant perturbation for g>1

6. Conclusions and further perspectives

a. BCFT approach: possibility of realizing Josephson devices with nontrivial phase diagrams;

b. Finite-coupling fixed-point based qubit: robust quantum coherence of the degenerate states (remarkable improvement with respect, e.g., rf-SQUIDS);

c. Effects of noise (D.G., P. Sodano, work in preparation);

d. More realistic models (interaction between chains and researvoirs: Affleck-Zagoskin-Caux model);

e. Efficient way to apply the phases φ_i (d-wave sc?).