Topological Quantum Computation with non-Abelian anyons in the Pfaffian quantum Hall state

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- **Acknowledegments:**
  Ivan Todorov, Ady Stern, Valentina Petkova, Chetan

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Not a big help on the scale of 500 qubits (huge overhead)

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• non-Abelian statistics might be easier to be observed than the Abelian one [Stern-Halperin, Kitaev et al.]

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- **Unpresedented precision** of quantum information processing

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- **Promise:** We shall only use universal quantities for TQC!

TQC scheme of Das Sarma et al.

- **Main idea:** use 4 q.h. Pfaffian wave functions as qubit (2-dim. space) N electrons (holes) at $z_1, \ldots, z_N$ and 4 quasiholes at $\eta_1, \ldots, \eta_4$

$$\Psi_{4\mathrm{qh}}(\eta_1, \eta_2, \eta_3, \eta_4; z_1, \ldots, z_N) \stackrel{\mathrm{def}}{=}$$

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$$\psi_{\mathrm{hole}}(z) = \psi(z) : \mathrm{e}^{i\sqrt{2}\phi(z)} : \quad \text{and} \quad \psi_{\mathrm{qh}}(\eta) = \sigma(\eta) : \mathrm{e}^{i\frac{1}{2\sqrt{2}}\phi(\eta)} :,$$

- **Explicit result [Nayak–Wilczek]:** $(\eta_{ab} = \eta_a - \eta_b)$

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Braiding: Extend TQC scheme of Das Sarma

- **Computational basis (reminder):**

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- **Fermion parity:** double degenrated Ramond sector $\sigma_\pm$

$$[\psi_0, \gamma_F]_+ = 0, \quad \psi_0^2 = \frac{1}{2}, \quad \gamma_F^2 = 1 \quad \Rightarrow \quad \gamma_F \sigma_\pm \gamma_F = \pm \sigma_\pm$$

- **Physical quasihole:** Modular invariance $\Rightarrow$ GSO projection $\Rightarrow$ non-Abelian statistics

$$\sigma(\eta) = \frac{\sigma_+(\eta) + \sigma_-(\eta)}{\sqrt{2}} \quad \Rightarrow \quad \sigma \times \sigma = \mathbb{I} + \psi$$

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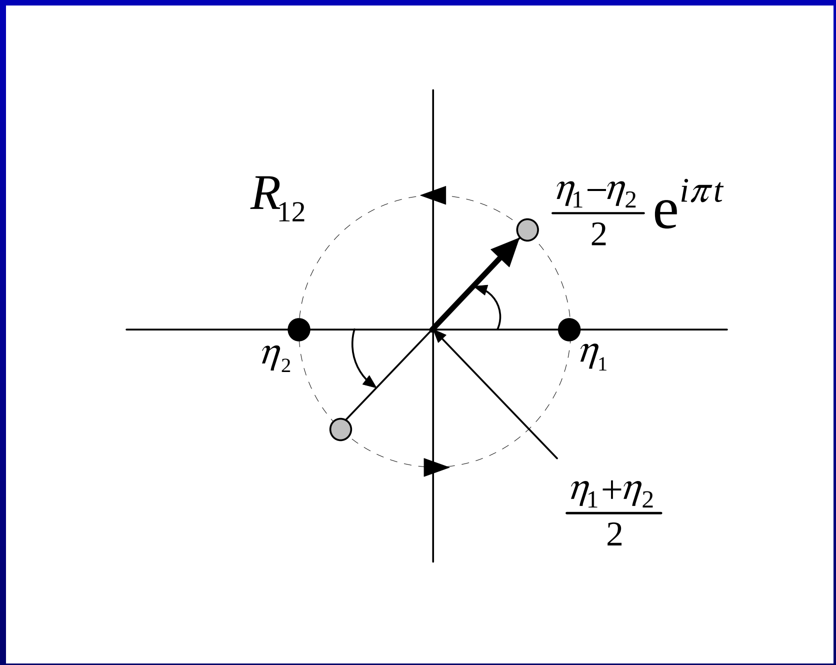
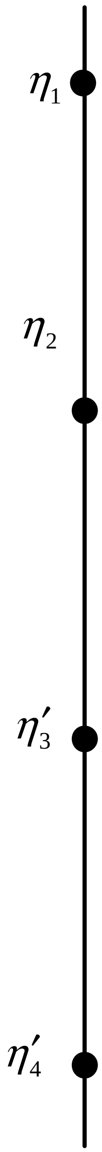
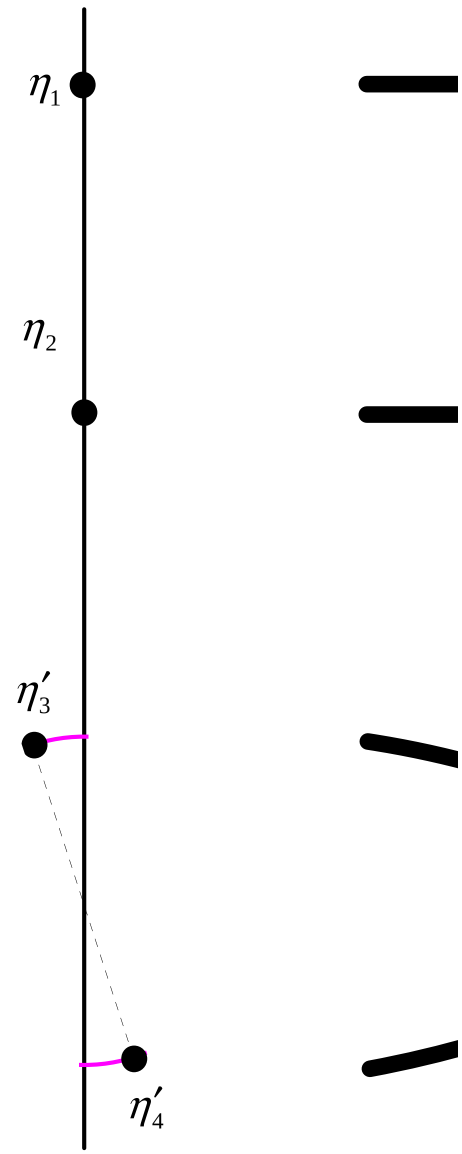
- **Braiding as analytic continuation:** $(0 \le t \le 1)$

$$\eta_1' = \frac{\eta_1 + \eta_2}{2} + e^{i\pi t}\frac{\eta_1 - \eta_2}{2}, \quad \eta_2' = \frac{\eta_1 + \eta_2}{2} - e^{i\pi t}\frac{\eta_1 - \eta_2}{2},$$




η_1
η_2
η_3'
η_4'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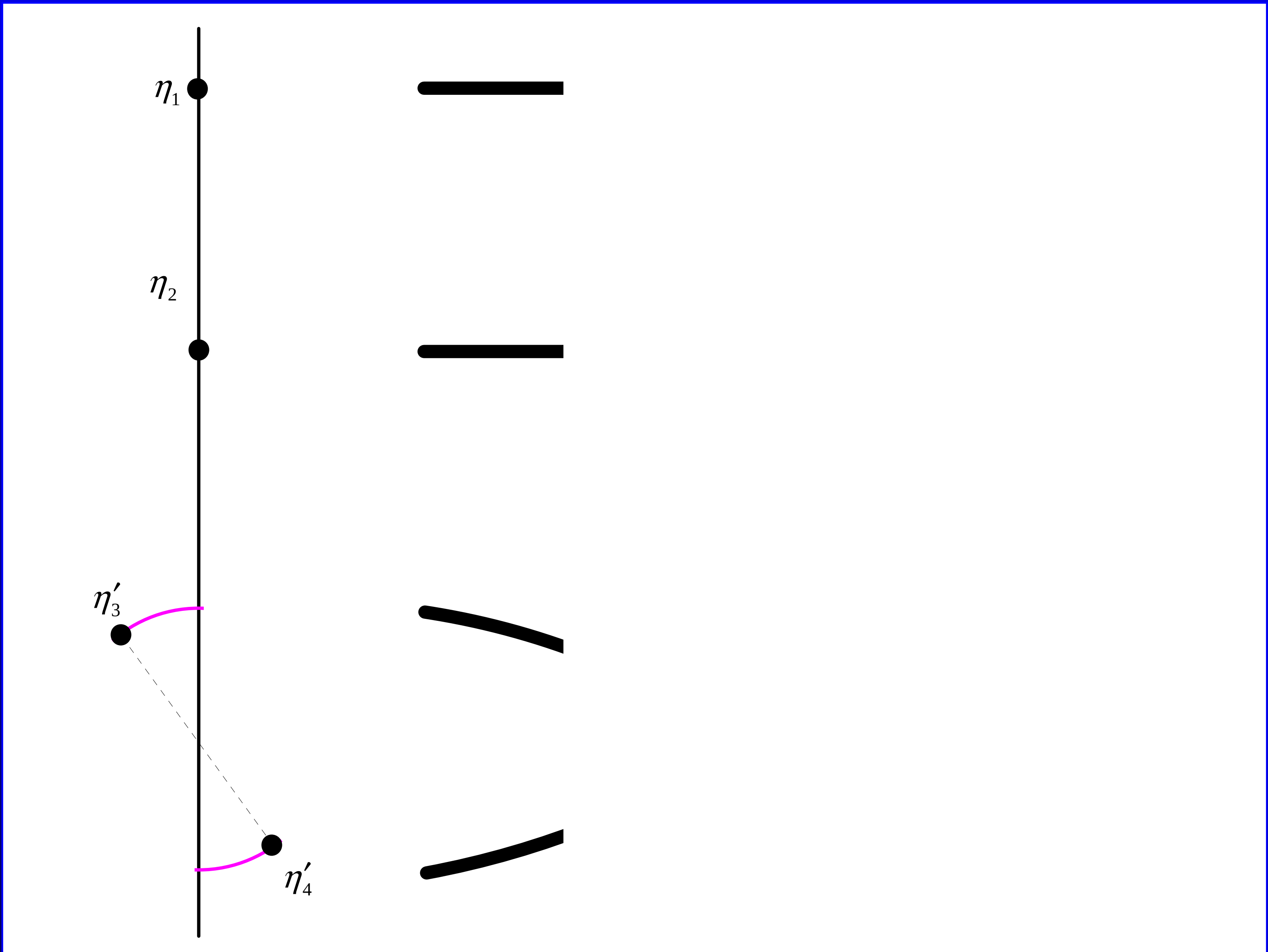

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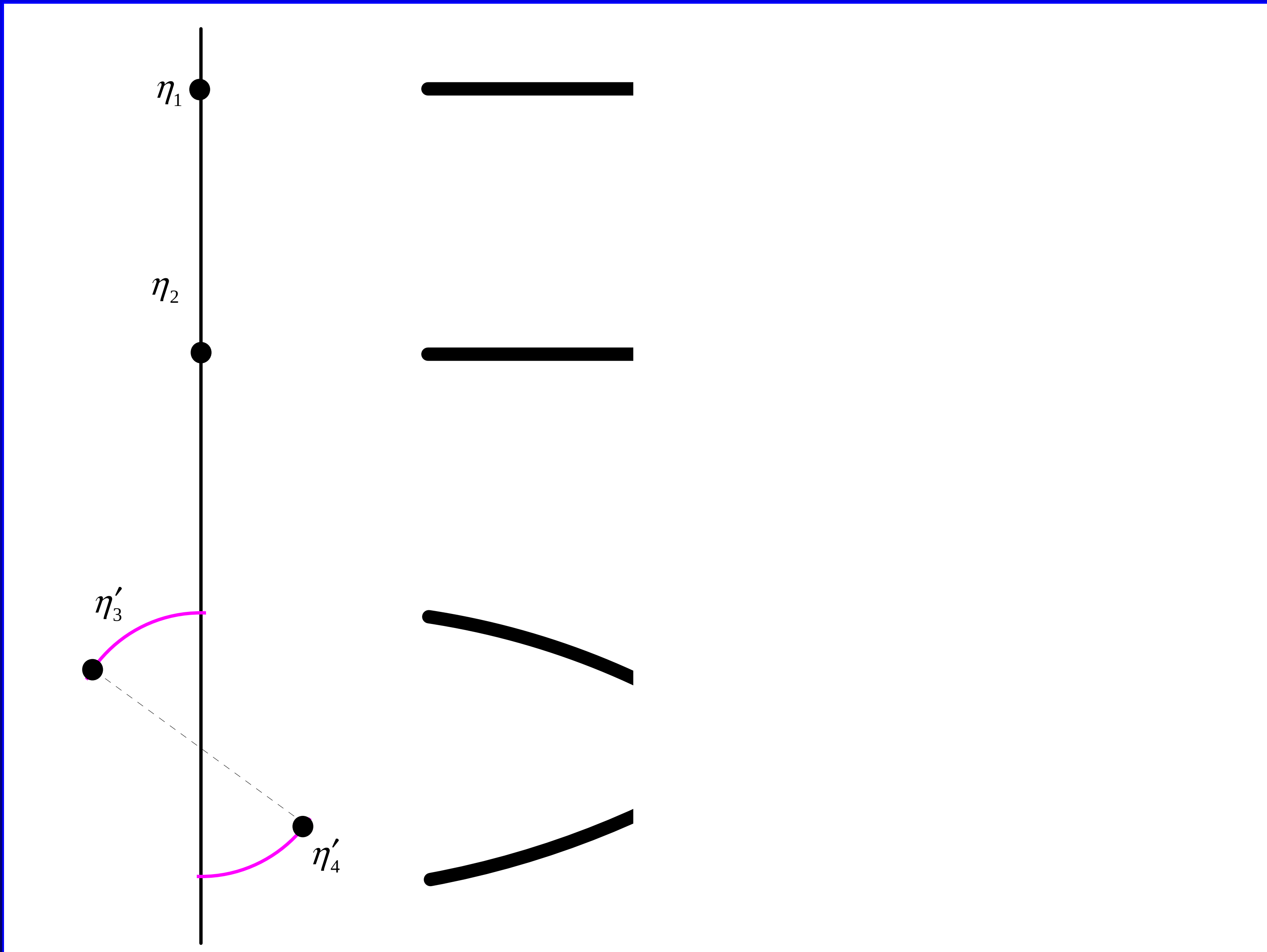
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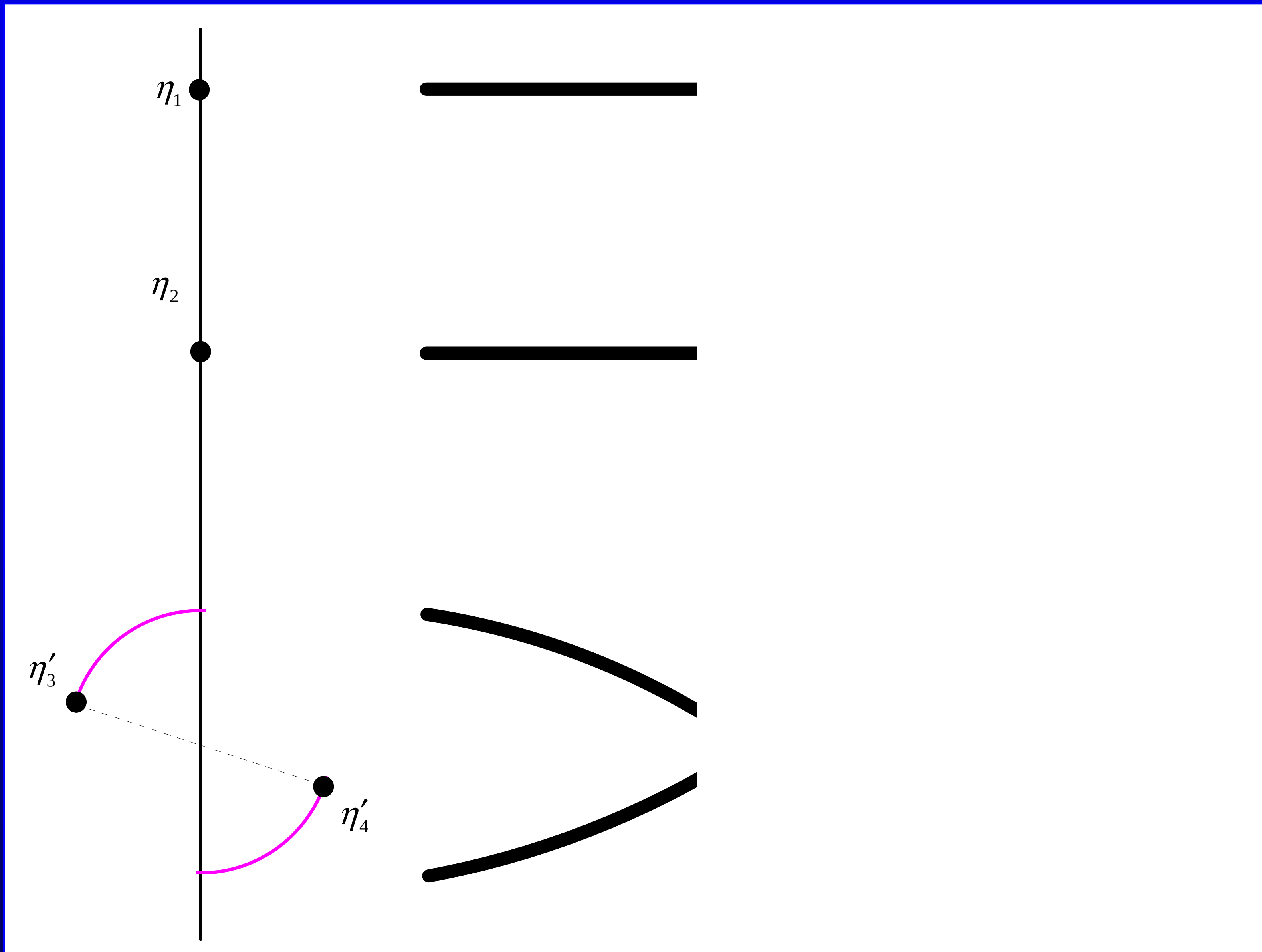
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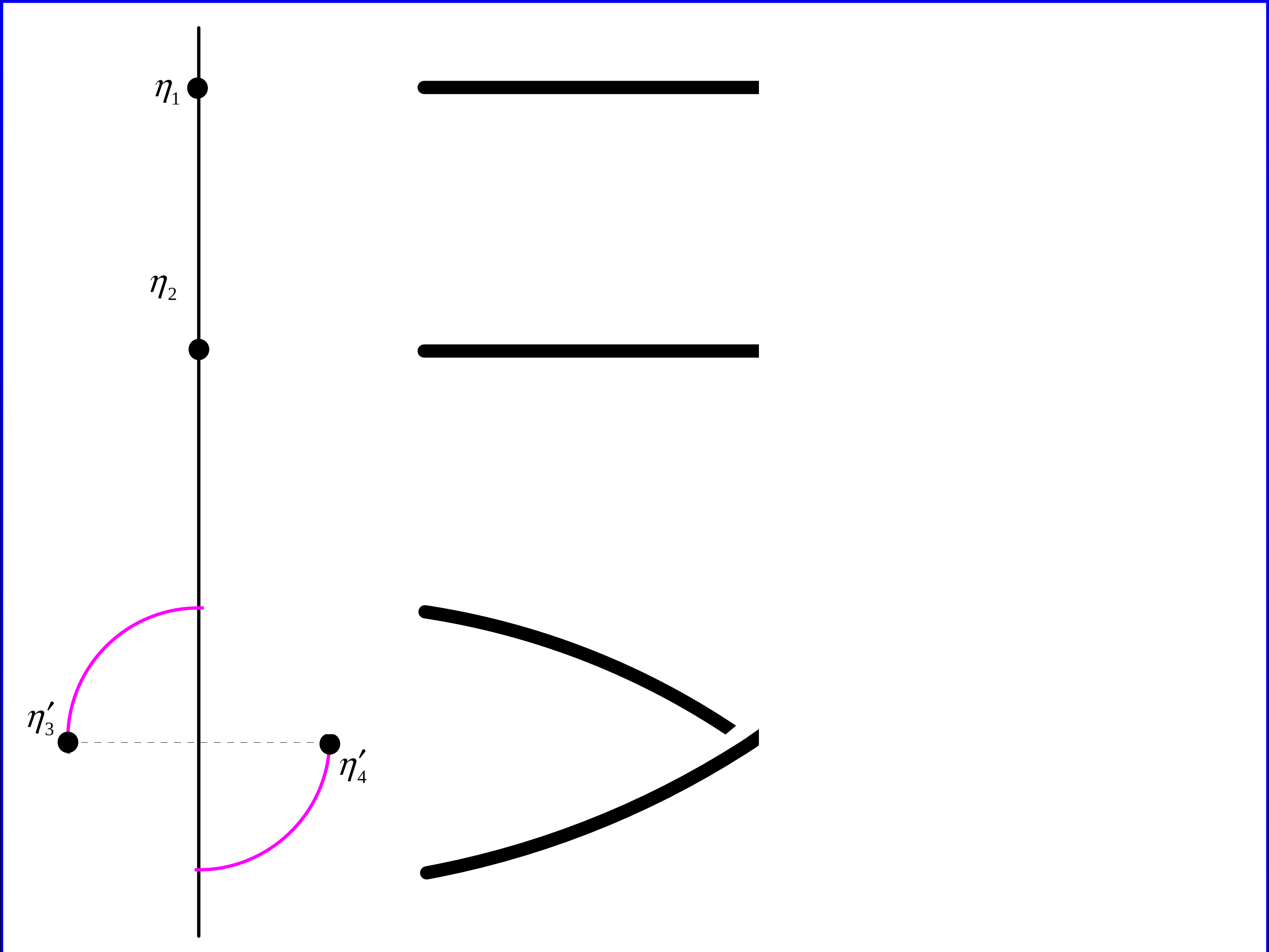
η_1
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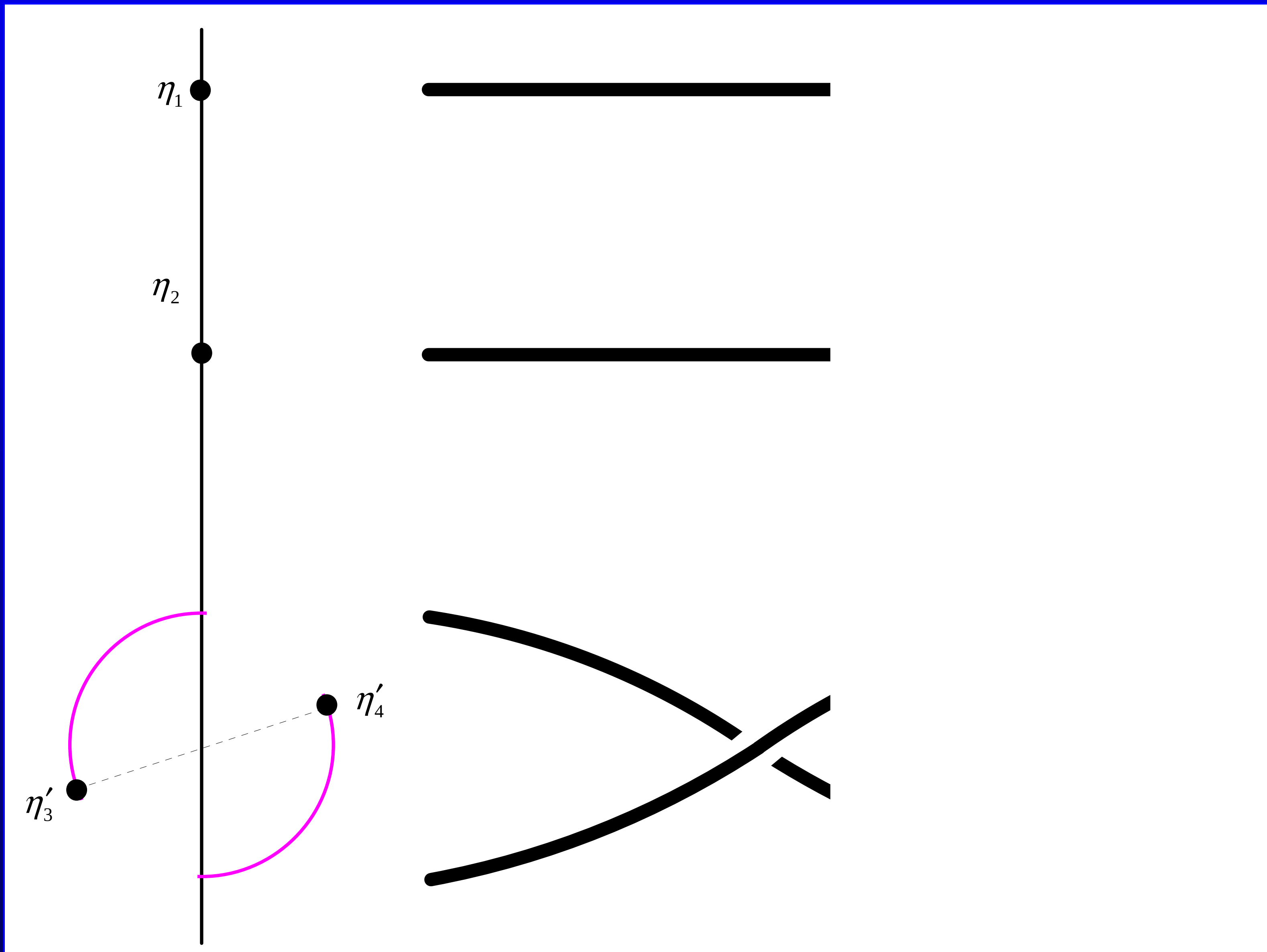
η_1
η_2
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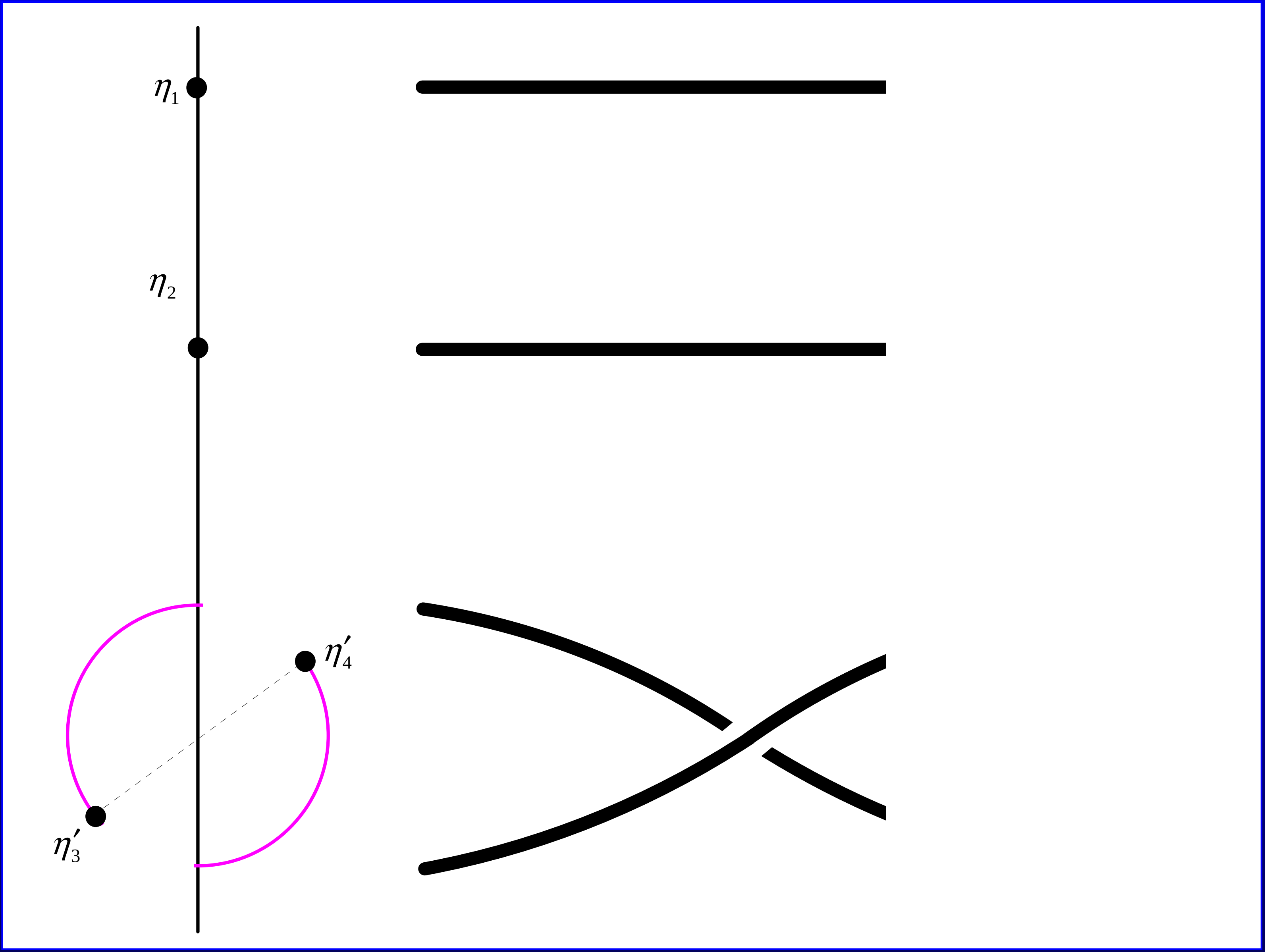
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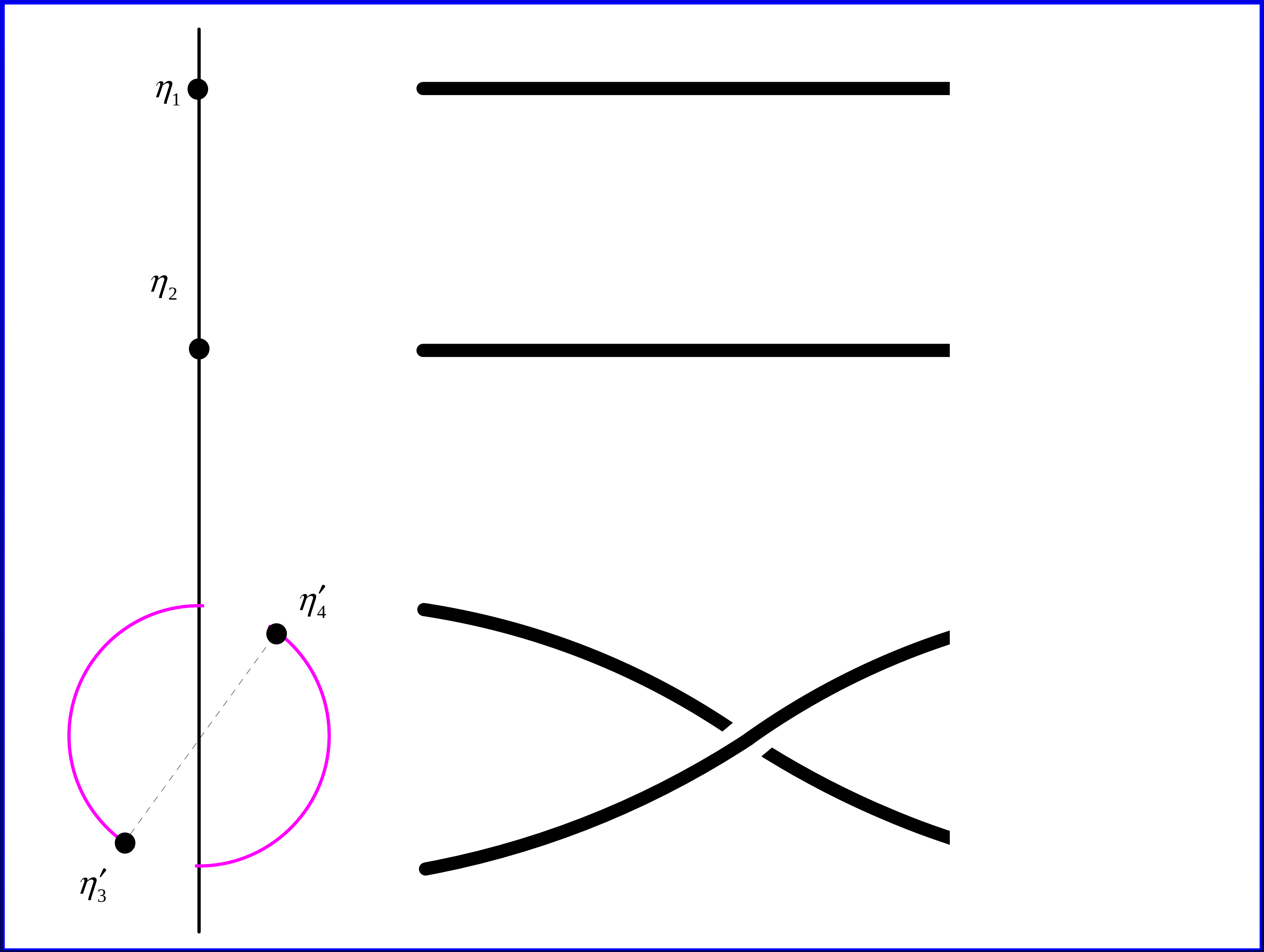
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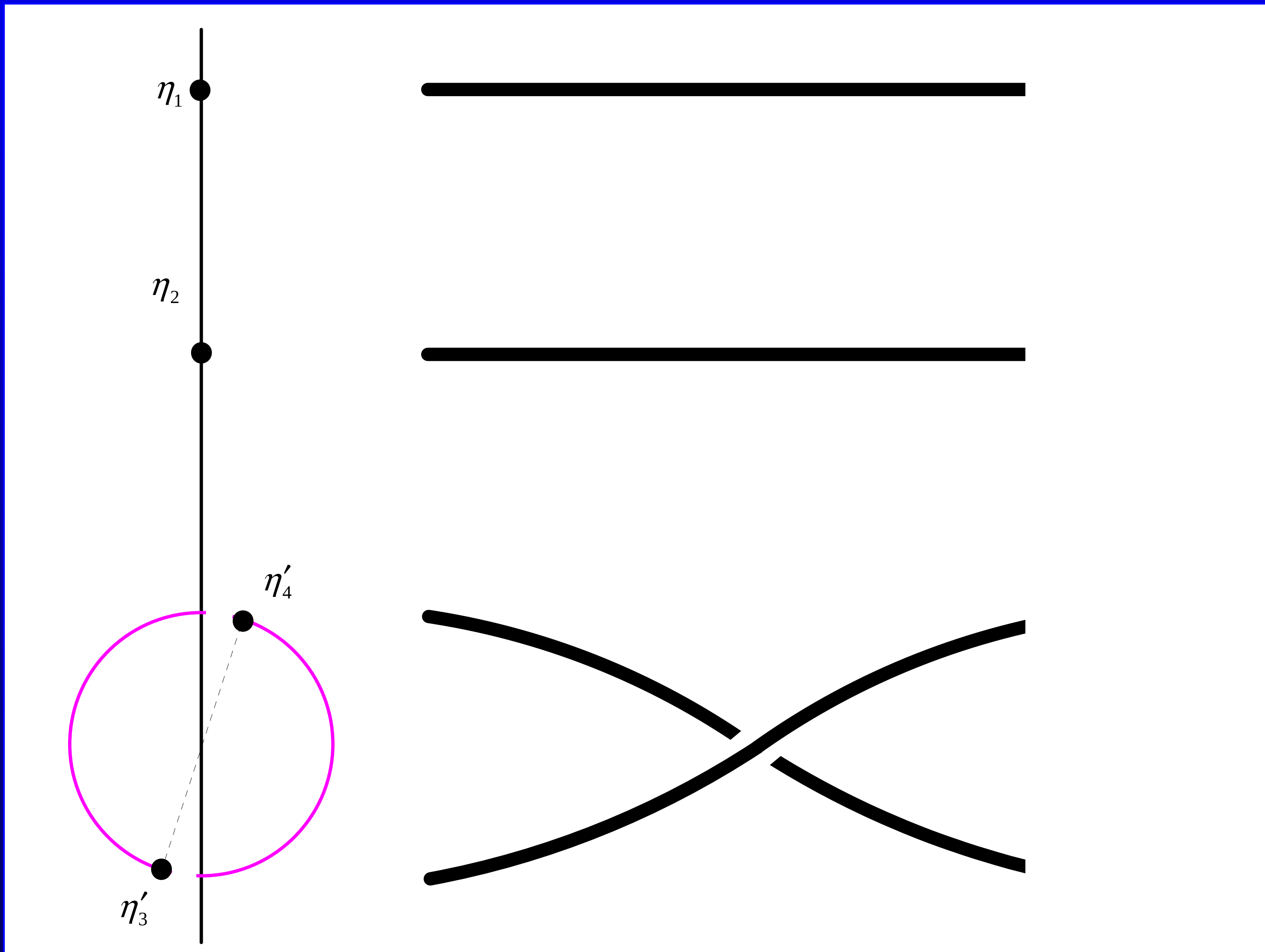
η_1
η_2
η_4'
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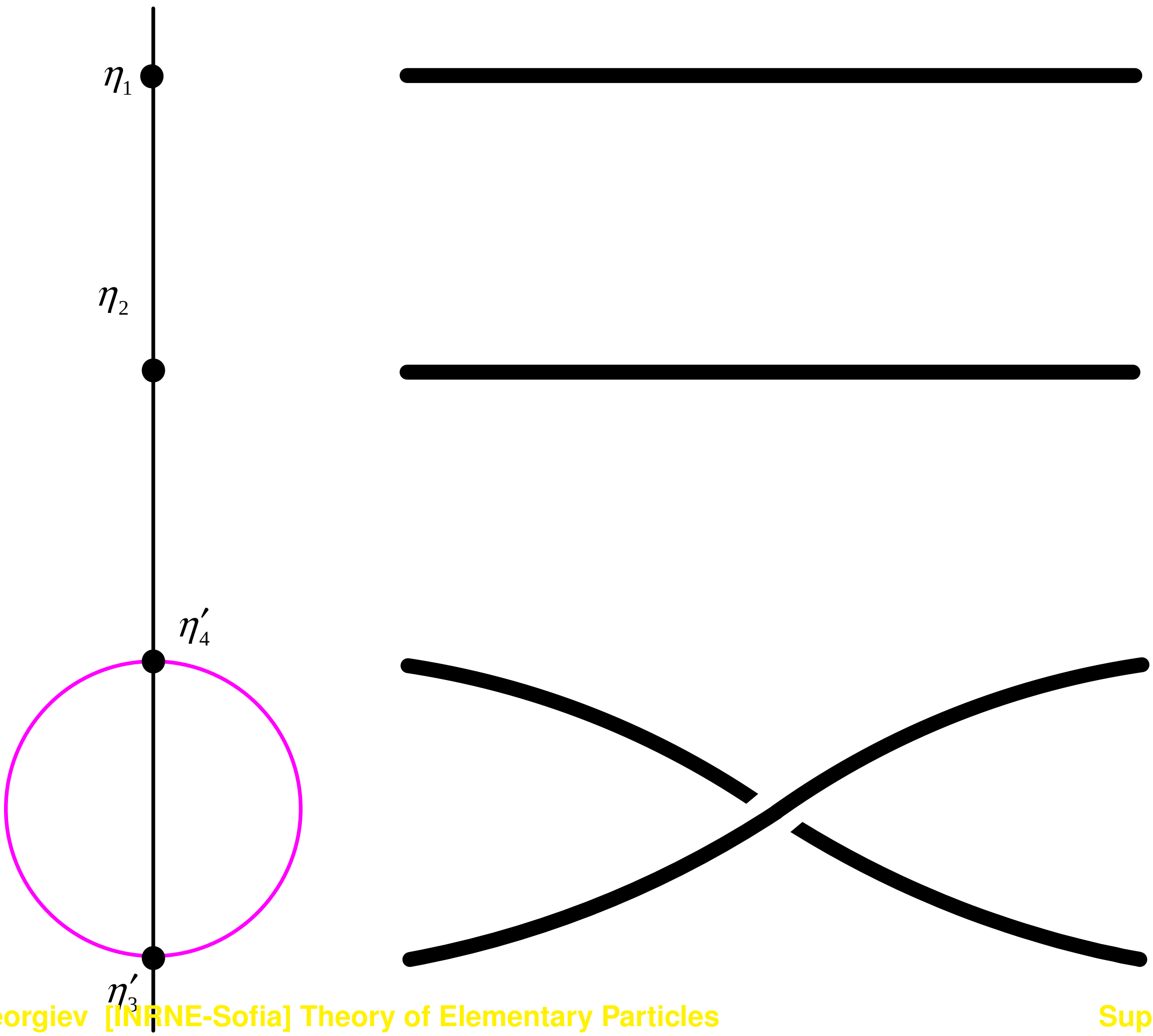
η_1
η_2
η'_4
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Exchange matrices for 4 Pfaffian quasiholes

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- Finite 2-dimensional representation of the braid group $\mathcal{B}_4$

$$R_{12}^{(4)} = R_{34}^{(4)} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad R_{23}^{(4)} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

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- Sign ambiguity resloved either by fusion consistency or by invoking Yang–Baxter equations

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$$R_{23}^{(4)} = R_{12}^{(4)} R_{13}^{(4)} \left(R_{12}^{(4)} \right)^{-1}$$

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$$\mathcal{D}_4 \equiv \{\pm\mathbb{I}_2,\ \pm\sigma_1,\ \pm i\sigma_2,\ \pm\sigma_3\},$$

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$$
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Dimino's algorithm: given the generators of a finite group, enumerate the group elements [Lou Dimino, Bell Labs. 1971]

For the group $\mathcal{B}_4$ Dimino's algorithm gives [Maple with modules]

$$|\mathrm{Image}\,(\mathcal{B}_4)\,| = 96.$$

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For general $2n \geq 6$ [Read]

$$|\mathrm{Image}\,(\mathcal{B}_{2n})\,| = \begin{cases} 2^{2n-1}(2n)! & \text{for} \quad n = \text{even} \\ 2^{2n}(2n)! & \text{for} \quad n = \text{odd} \end{cases}.$$

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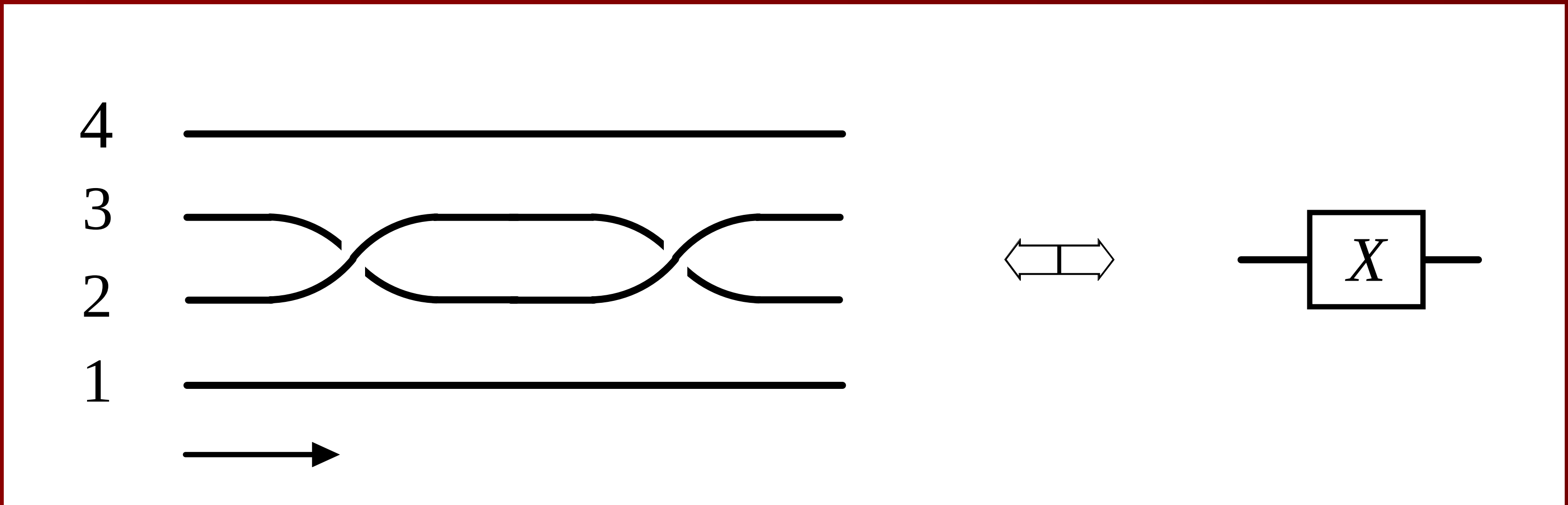
• **Too bad for TQC:** for universal TQC we need $\mathrm{Image}\,(\mathcal{B}_{2n})$ to be dense in the unitary group

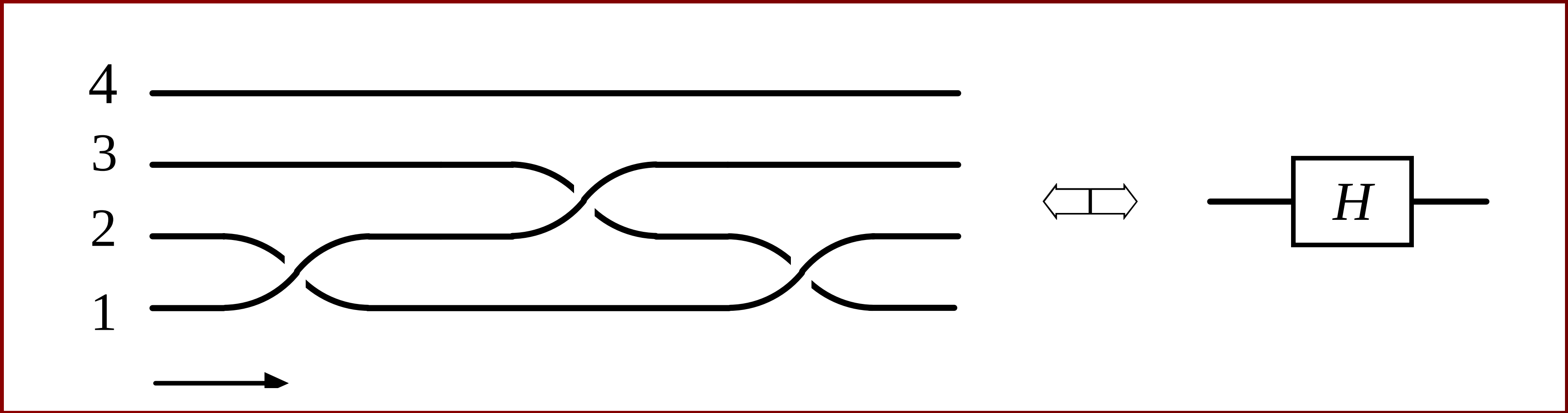
Single-qubit gates: The Pauli X gate

The NOT gate [Das Sarma et al.] $X \equiv R_{23}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



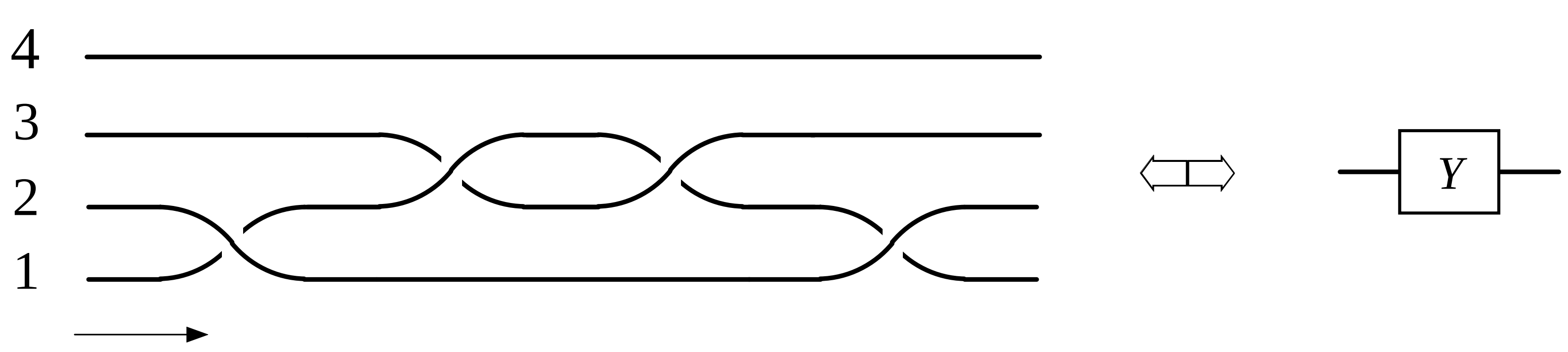
Single-qubit gates: the Hadamard gate

$$H \simeq R_{12}^2 R_{13} = R_{12} R_{23} R_{12} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$



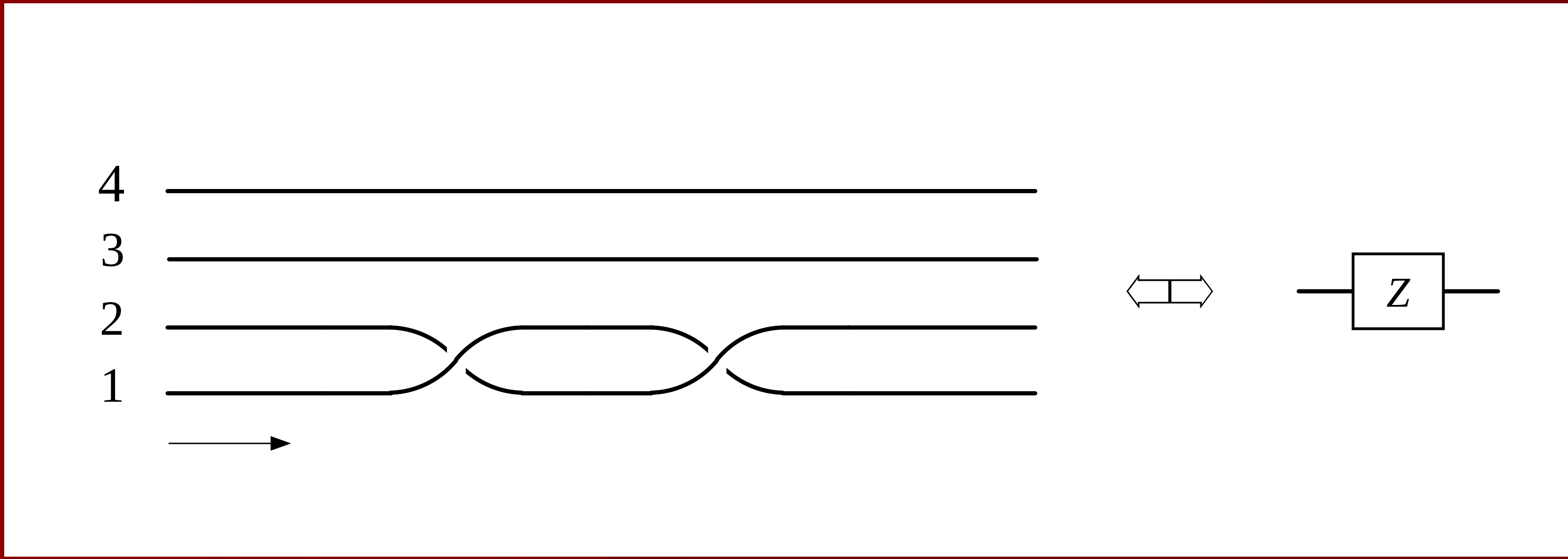
Single-qubit gates: the Pauli Y gate

(defined in QC without i) $Y \equiv R_{12}^{-1} R_{23}^{2} R_{12} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



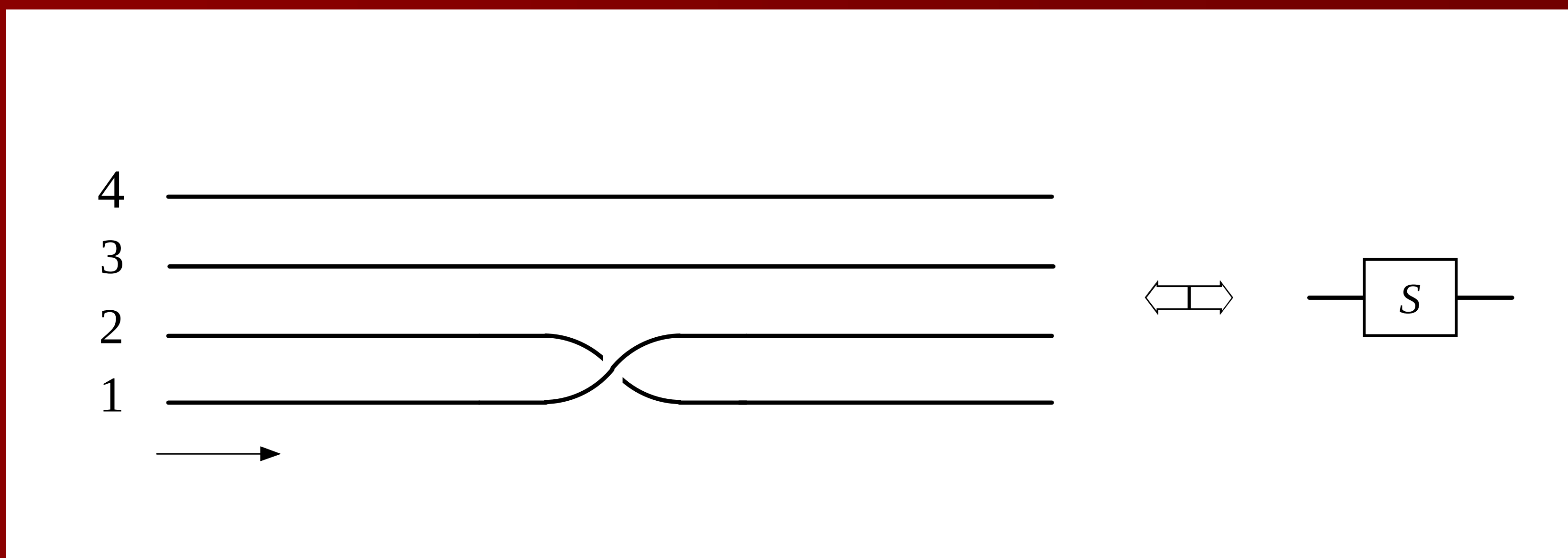
Single-qubit gates: the Pauli Z gate

$$Z \equiv R_{12}^2 = R_{34}^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Single-qubit gates: the phase gate S

$$S \equiv R_{12} = R_{34} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$



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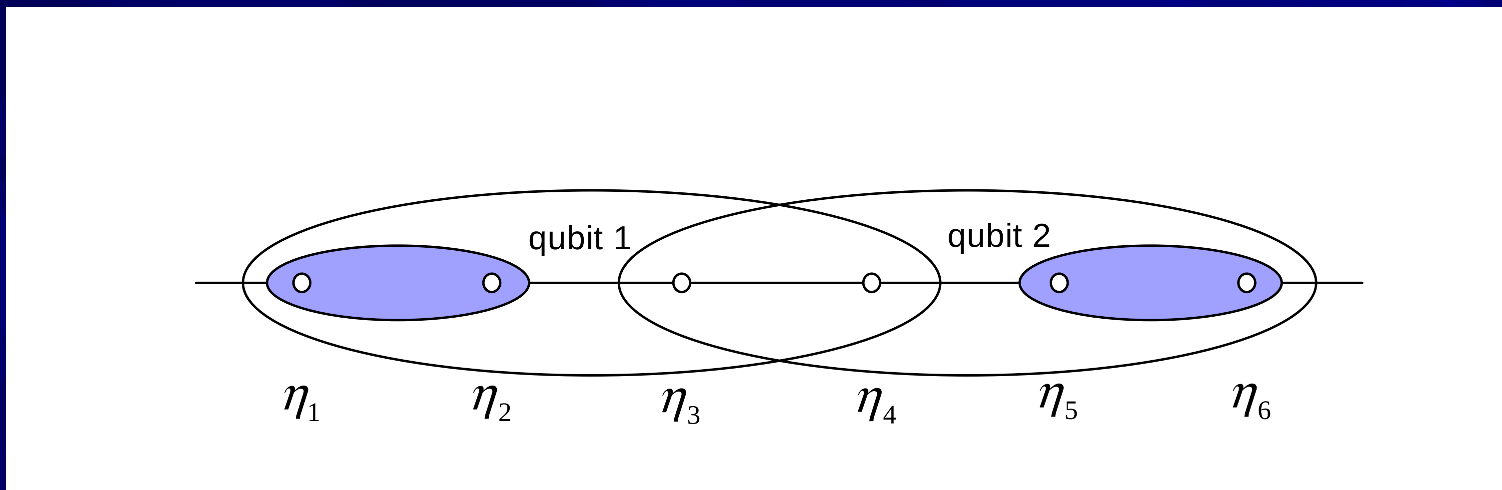

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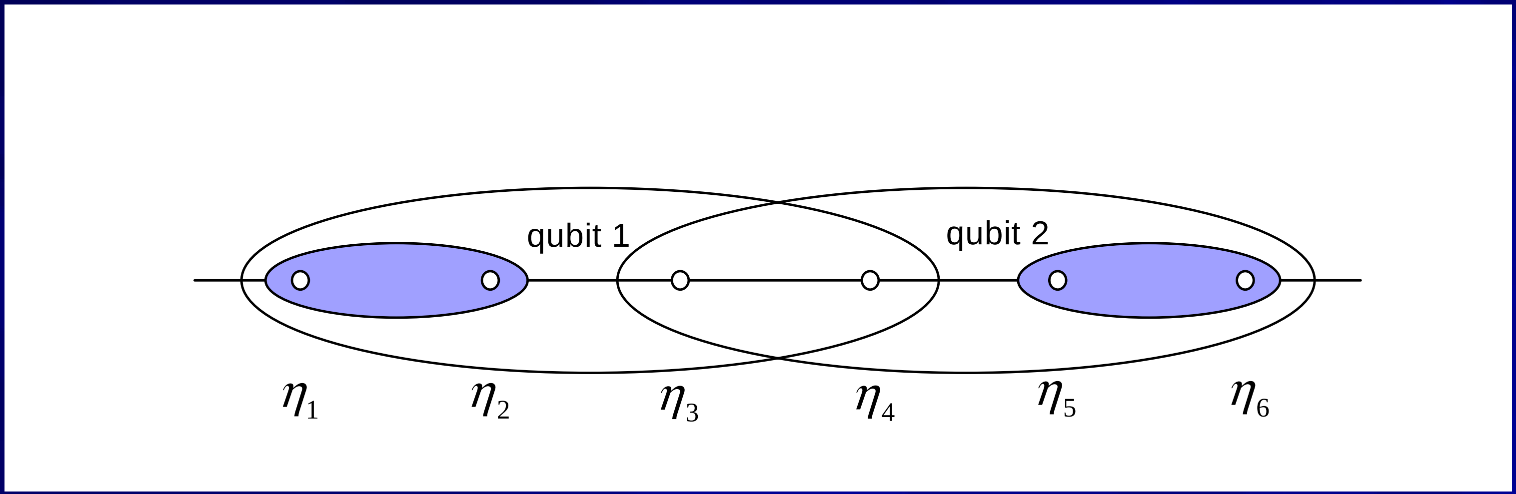

$$|00\rangle \equiv \langle\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\rangle, \quad |01\rangle \equiv \langle\sigma_+\sigma_+\sigma_+\sigma_-\sigma_+\sigma_-\rangle$$

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$$R_{12}^{(6)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} = R_{12}^{(4)} \otimes \mathbb{I}_2.$$

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$$R_{56}^{(6)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} = \mathbb{I}_2 \otimes R_{34}^{(4)}$$

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- **Doesn't work for** $R_{34}^{(6)}$ (not factorizable)
- **However:** $R_{34}^{(6)}$ diagonal due to NS-sector supersel. rule
- $\Rightarrow$ Obtain $R_{34}^{(6)}$ directly from the OPE

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Single-qubit gates in the two-qubit basis

- **Hadamard gates acting on first and second qubits:**

$$H_1 \simeq H \otimes \mathbb{I}_2 \;=\; \left(R_{12}^{(6)}\right)^{-1} \left(R_{23}^{(6)}\right)^{-1} \left(R_{12}^{(6)}\right)^{-1} = \\ = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

$$H_2 \simeq \mathbb{I}_2 \otimes H = R_{56}^{(6)} R_{45}^{(6)} R_{56}^{(6)} = \frac{\mathrm{e}^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

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$$S_1 = S \otimes \mathbb{I}_2 = R^{(6)}_{12}$$

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- **Phase gates on first and second qubits:**

$$S_1 = S \otimes \mathbb{I}_2 = R^{(6)}_{12} \quad \text{and} \quad S_2 = \mathbb{I}_2 \otimes S = R^{(6)}_{56}.$$

Controlled-Z and Controlled-NOT gates

- **General construction:** [Nielsen–Chuang]

$$\mathrm{CZ} = \mathrm{e}^{i\frac{\pi}{4}}\mathrm{e}^{i\frac{\pi}{4}Z_1Z_2}\mathrm{e}^{-i\frac{\pi}{4}Z_1}\mathrm{e}^{-i\frac{\pi}{4}Z_2},$$

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- **Recall:**

$$\mathrm{e}^{\pm i\frac{\pi}{4} A} = \cos\left(\frac{\pi}{4}\right) \mathbb{I}_2 \pm i\, A \sin\left(\frac{\pi}{4}\right), \quad \text{for} \quad A^2 = \mathbb{I}_2$$

- **Observe:**

$$
\begin{aligned}
e^{i\frac{\pi}{4}Z_1 Z_2} &= e^{i\frac{\pi}{4}} \left(R_{34}^{(6)}\right)^{\dagger}, \\
e^{-i\frac{\pi}{4}Z_1} &= e^{-i\frac{\pi}{4}} R_{12}^{(6)}, \quad e^{-i\frac{\pi}{4}Z_2} = e^{-i\frac{\pi}{4}} R_{56}^{(6)}
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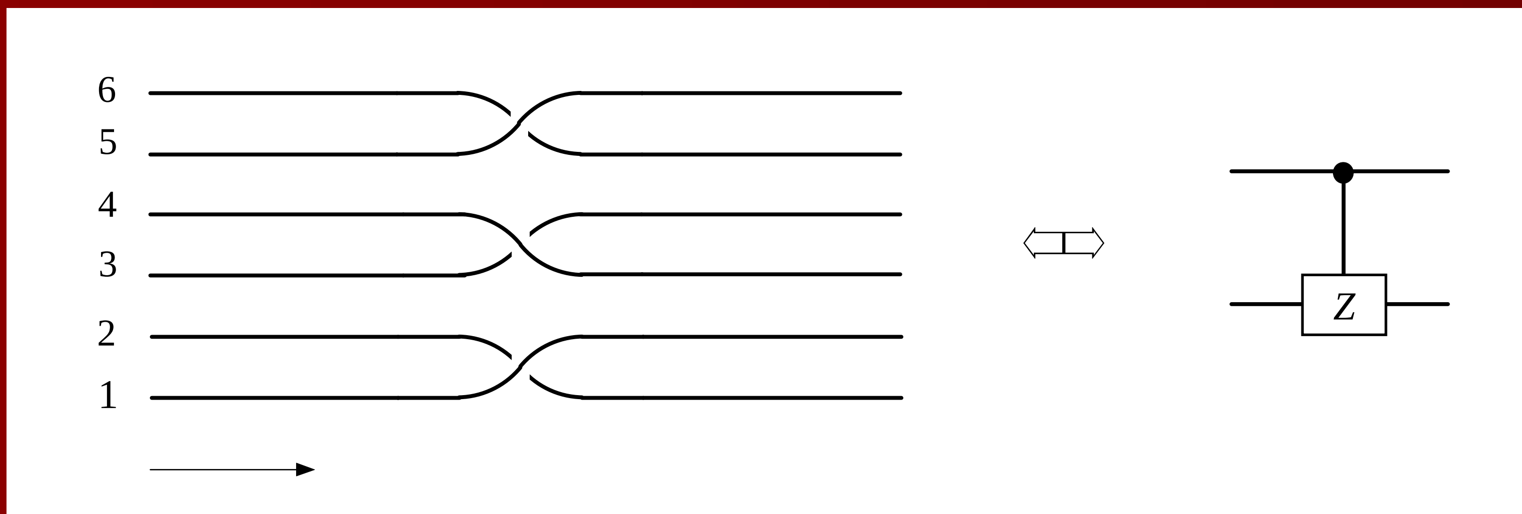
- **Therefore:**

$$\mathrm{CZ} = R_{12}^{(6)} \left(R_{34}^{(6)}\right)^{-1} R_{56}^{(6)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- **Braid diagram for CZ:**



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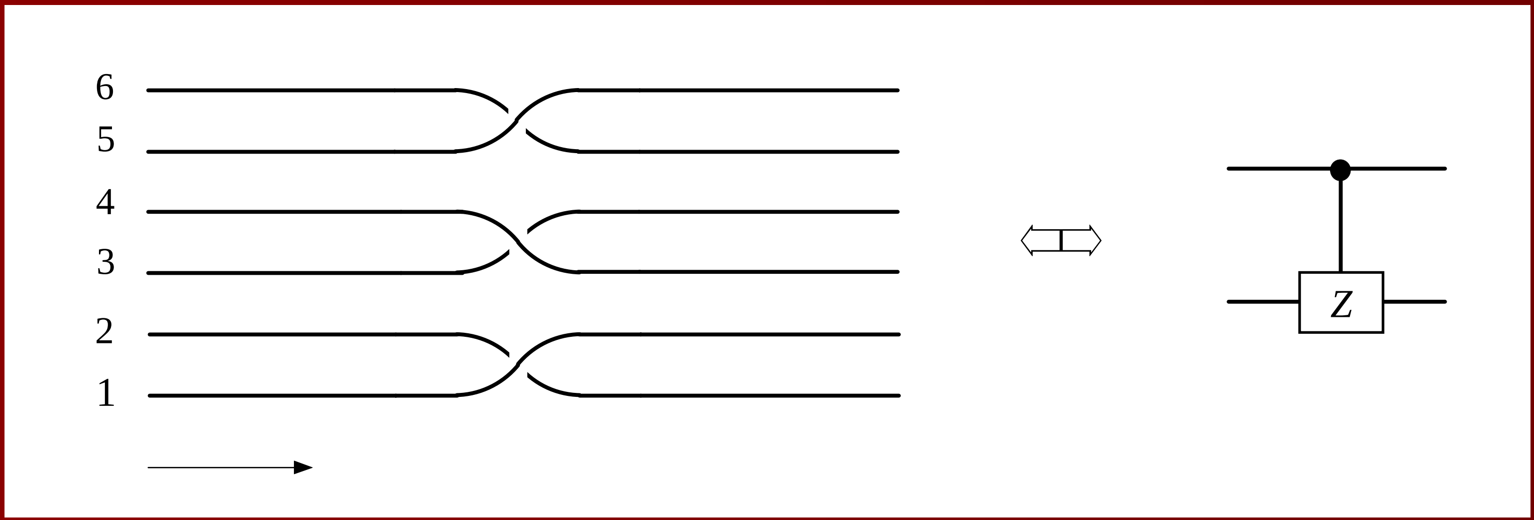

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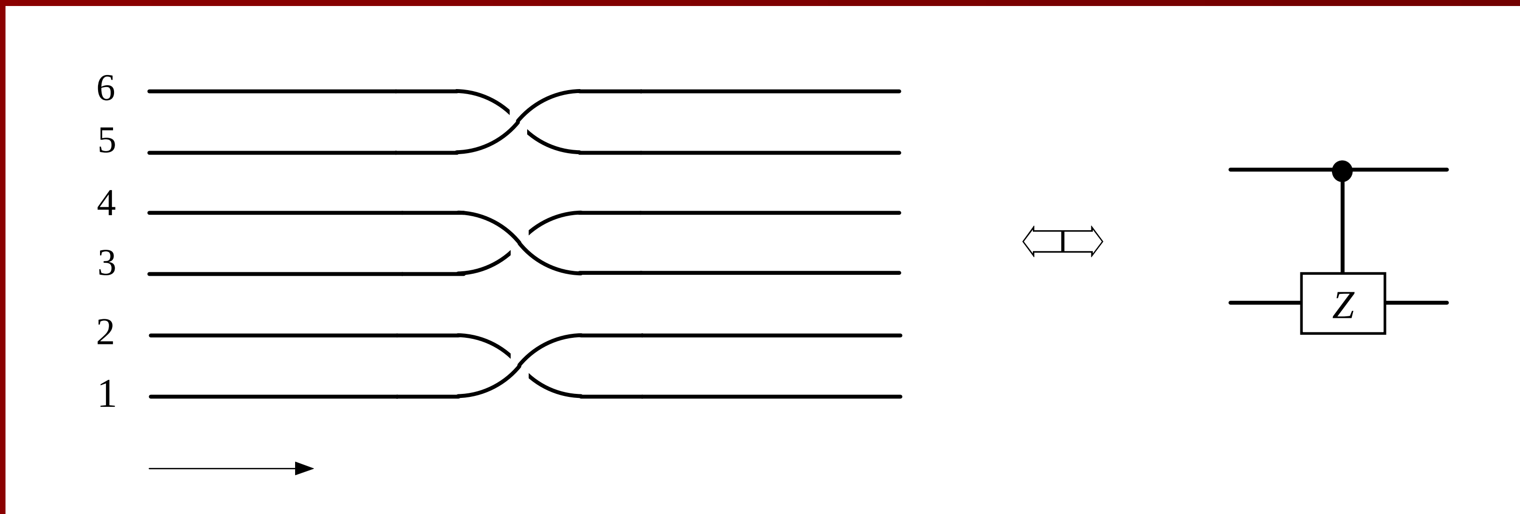

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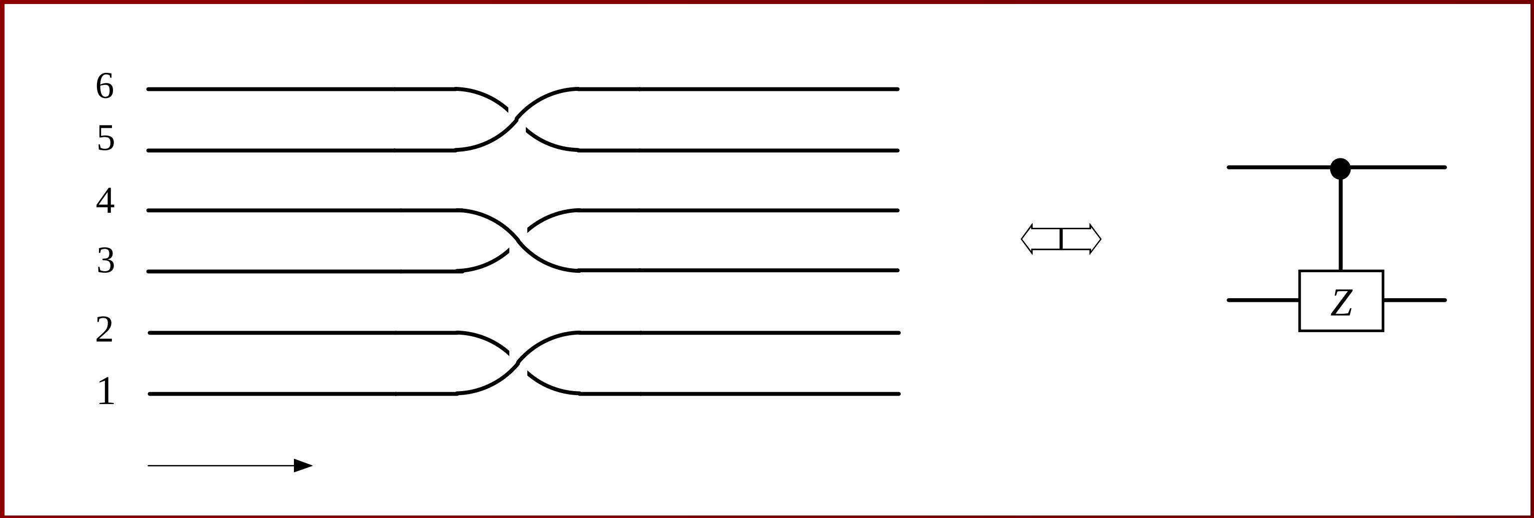

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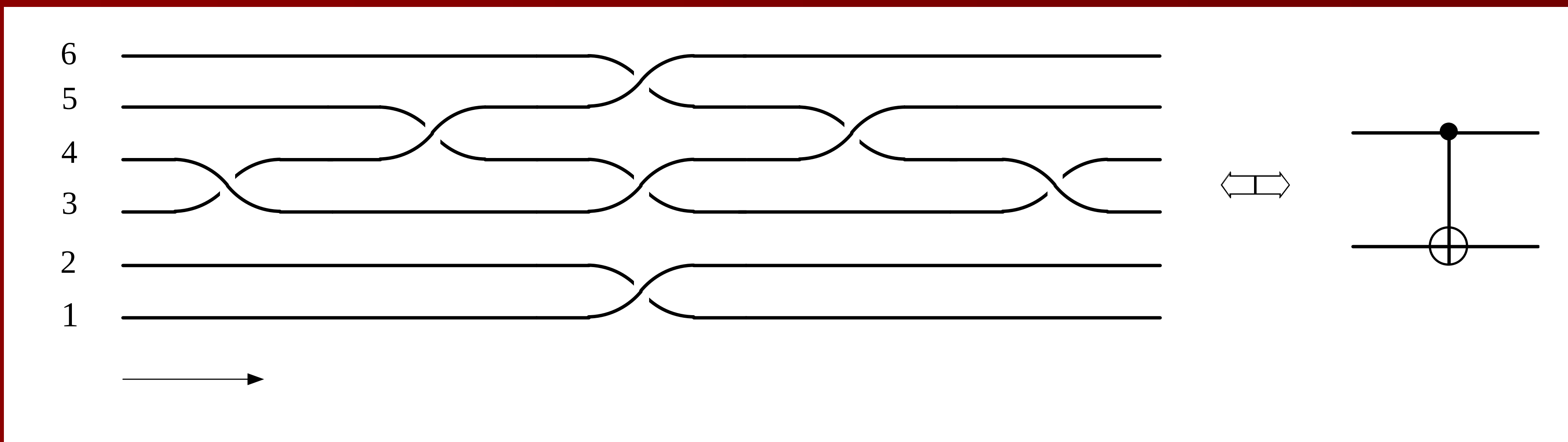
$\Longleftarrow$ Topological entanglement built into $R_{34}^{(6)}$

- **CNOT is now straightforward**

$$\mathrm{CNOT} = H_2\,\mathrm{CZ}\,H_2 = R_{56}R_{45}R_{56}^{-1}R_{34}^{-1}R_{12}R_{45}R_{56} \simeq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- **Alternatively**

$$\mathrm{CNOT} = R_{34}^{-1} R_{45} R_{34} R_{12} R_{56} R_{45} R_{34}^{-1} .$$



- **N.B.: 100 % topological protection for CNOT**
- **Just 7 elementary braids!**

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Exchange two neighbouring qubits (answer to S. Bravyi)

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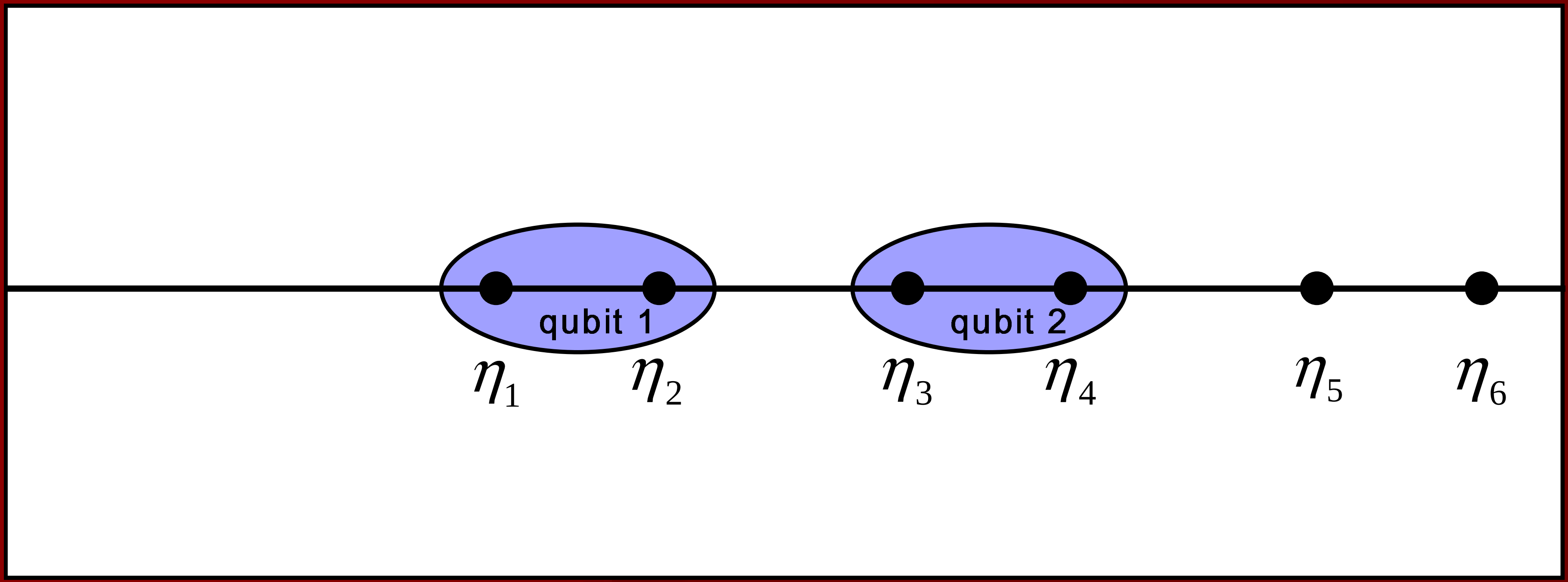

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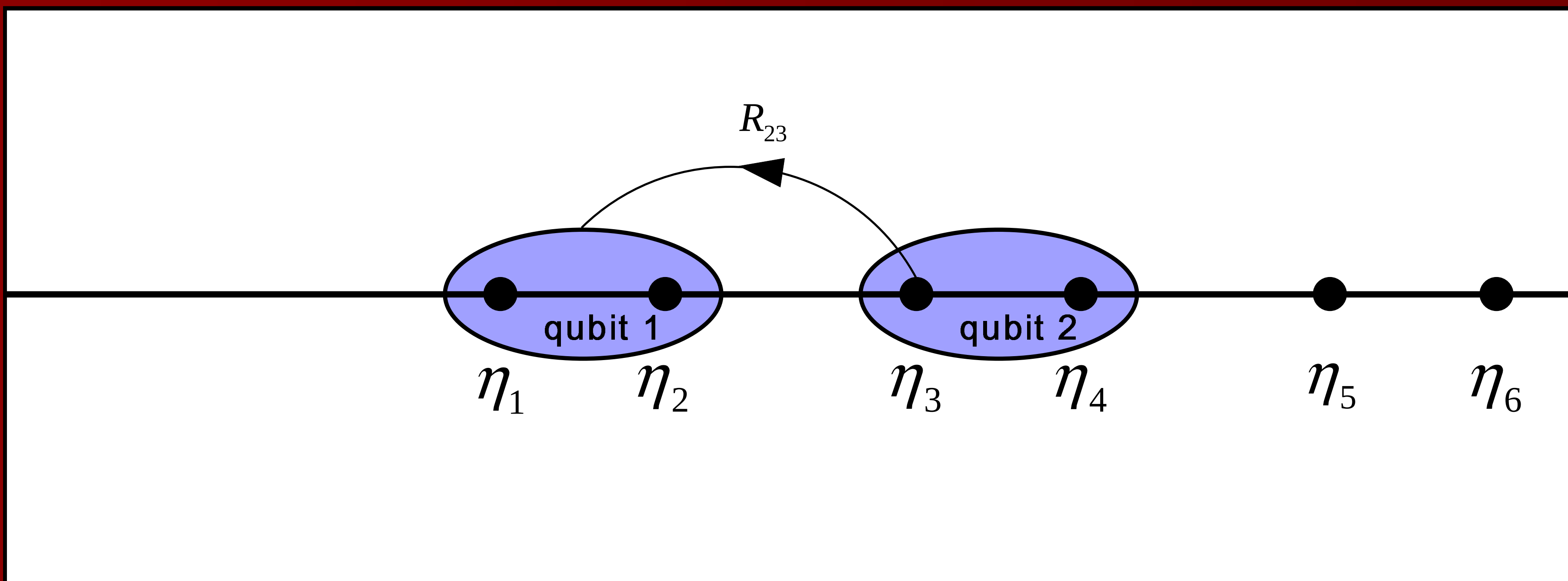

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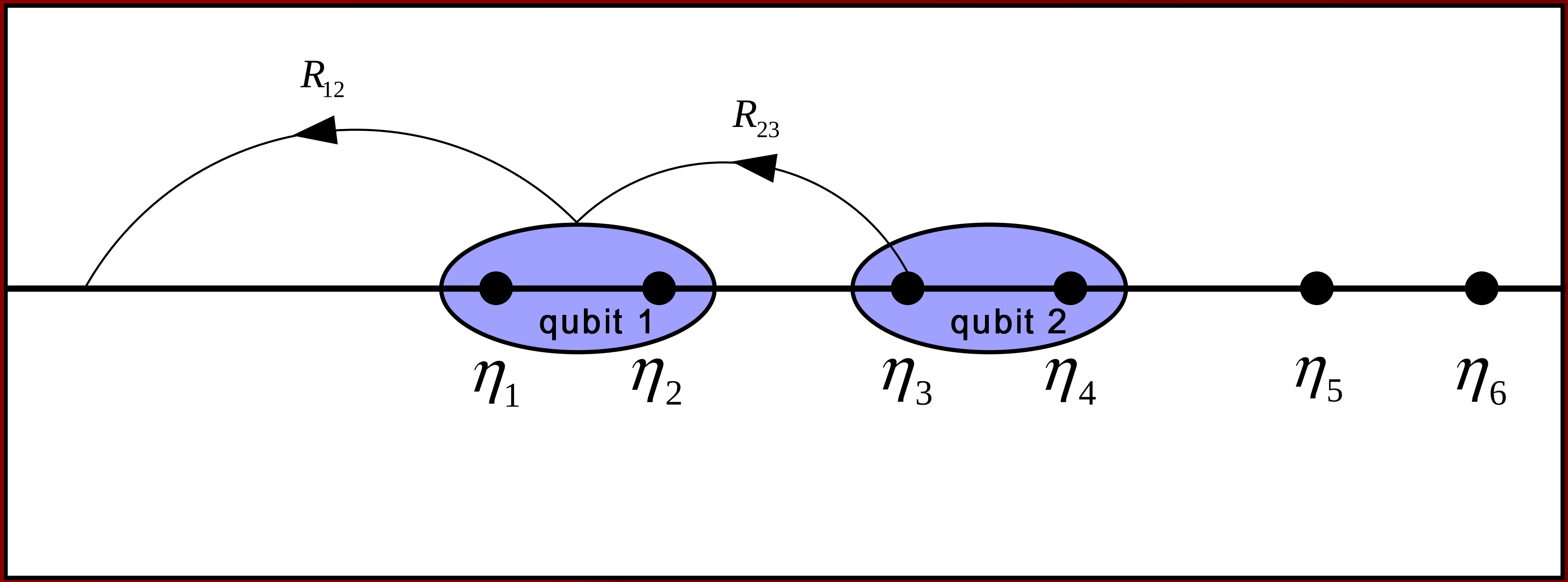

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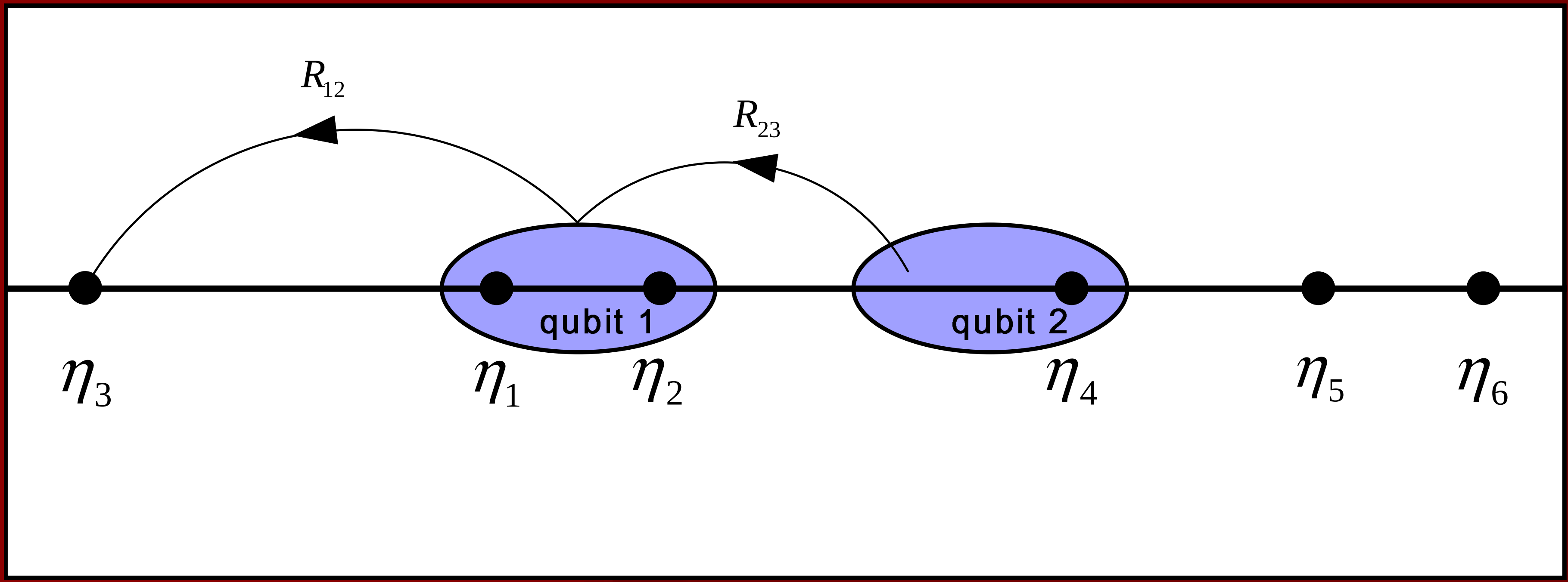

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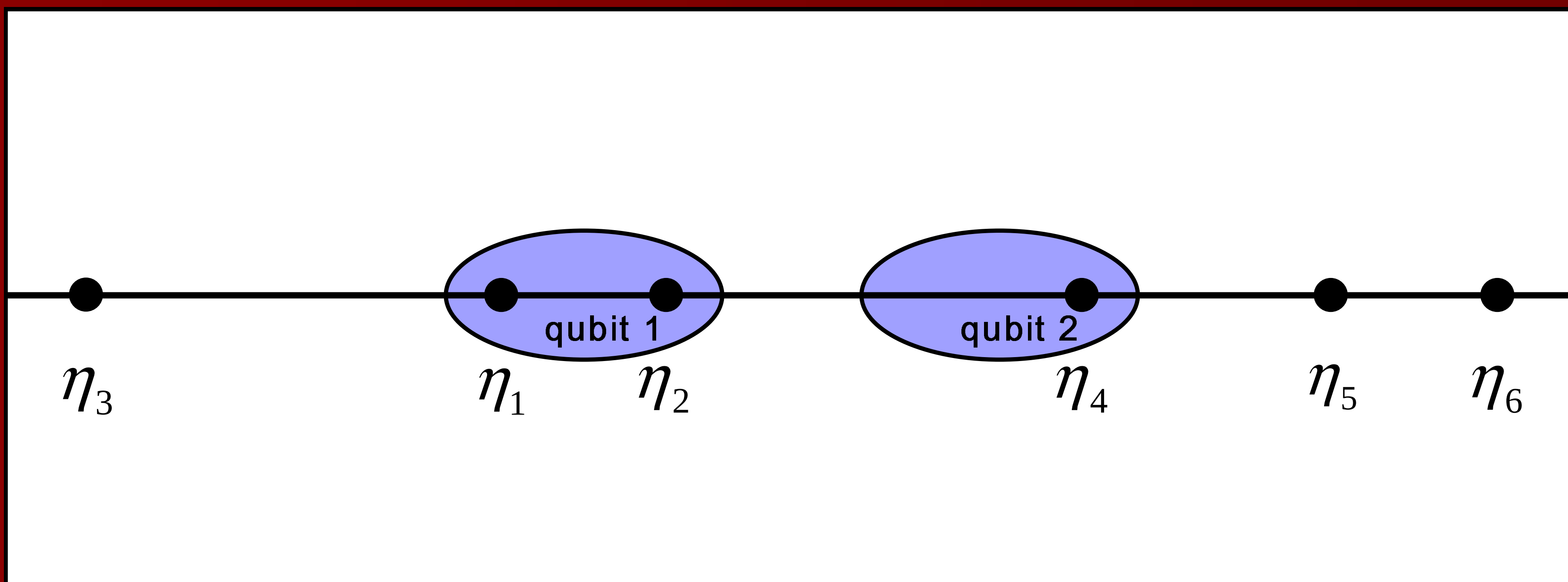

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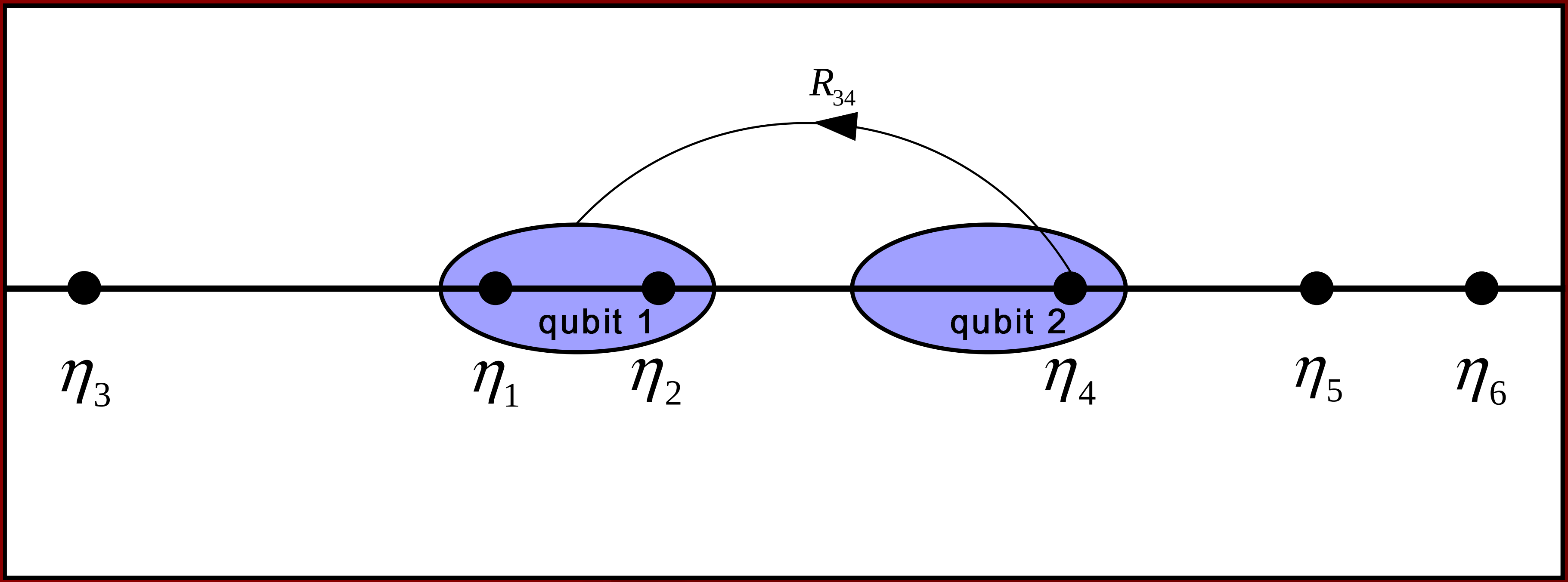

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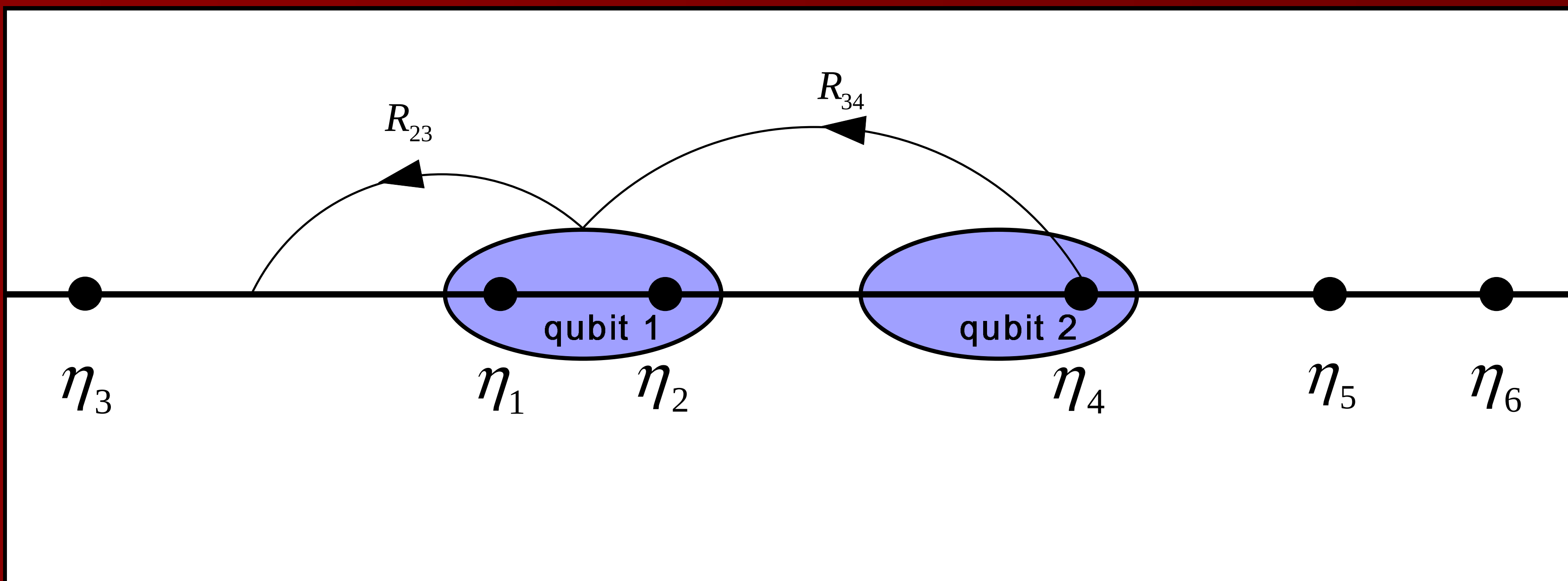

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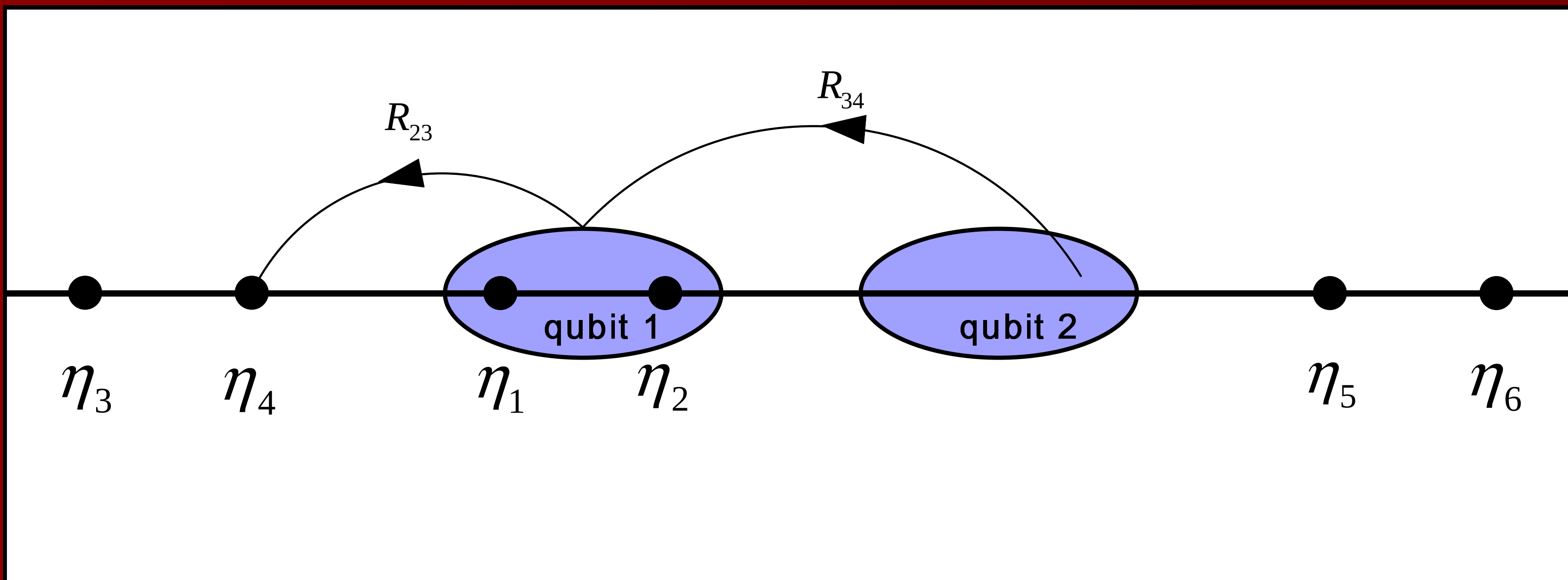

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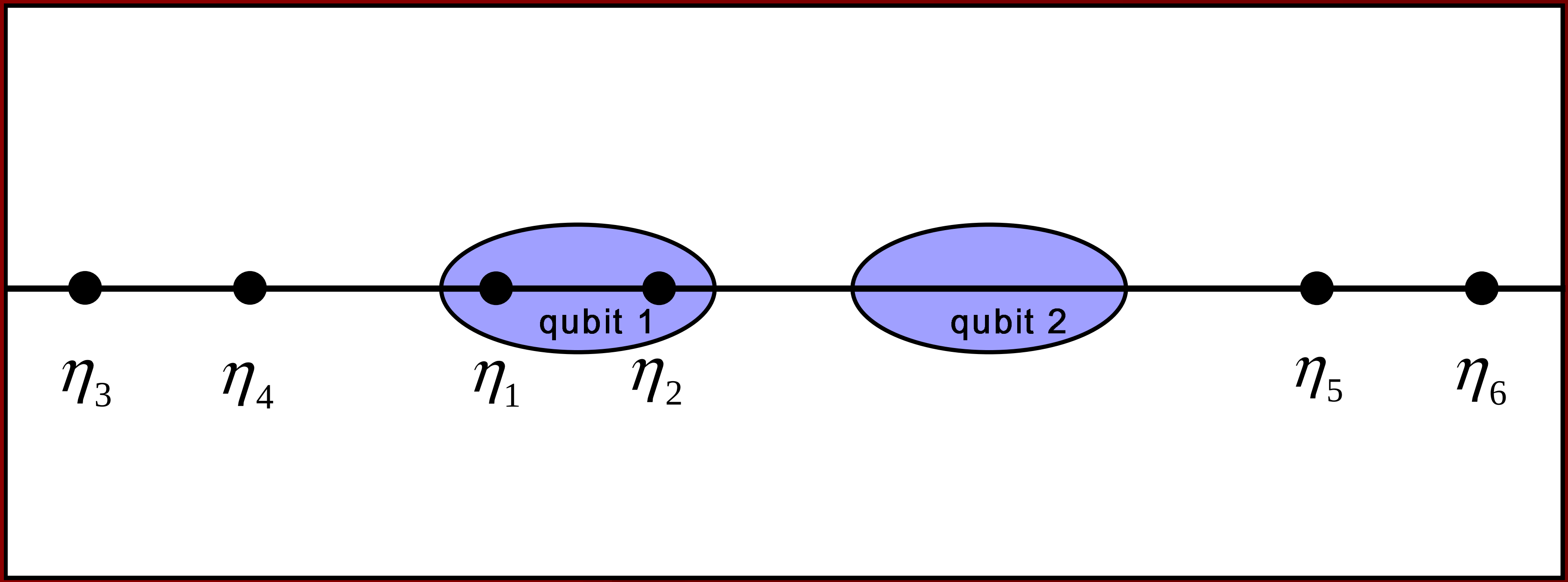

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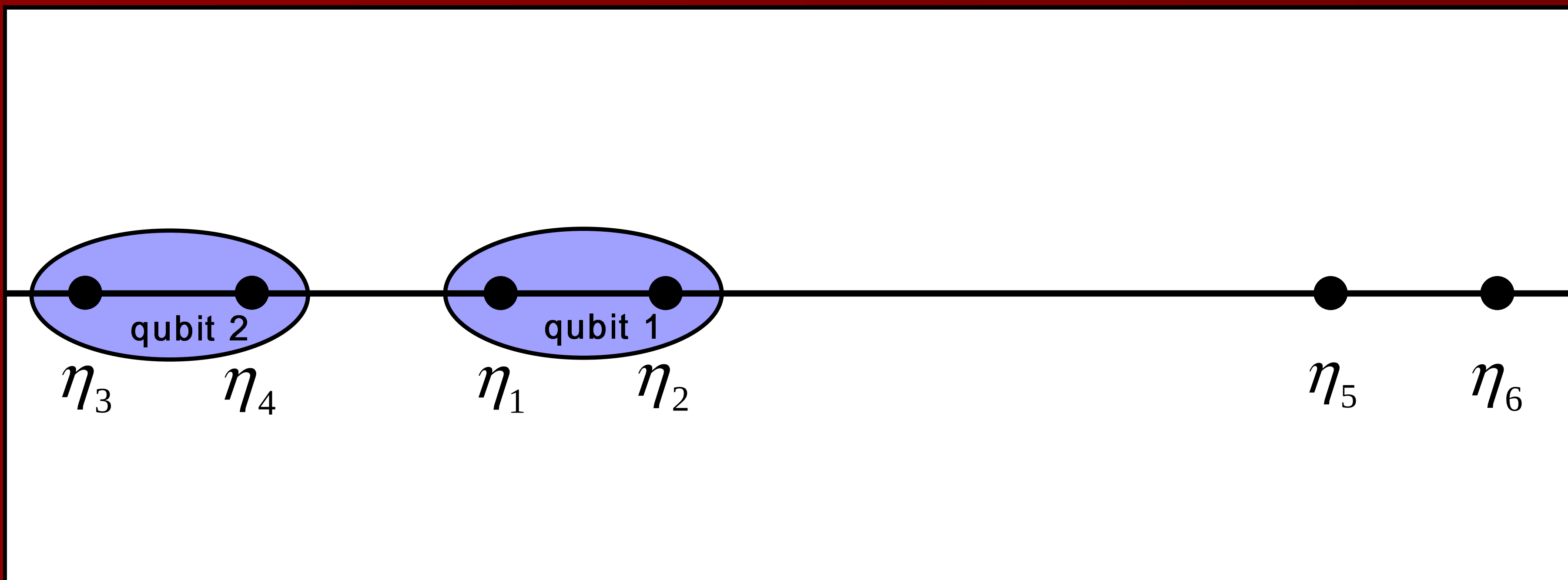

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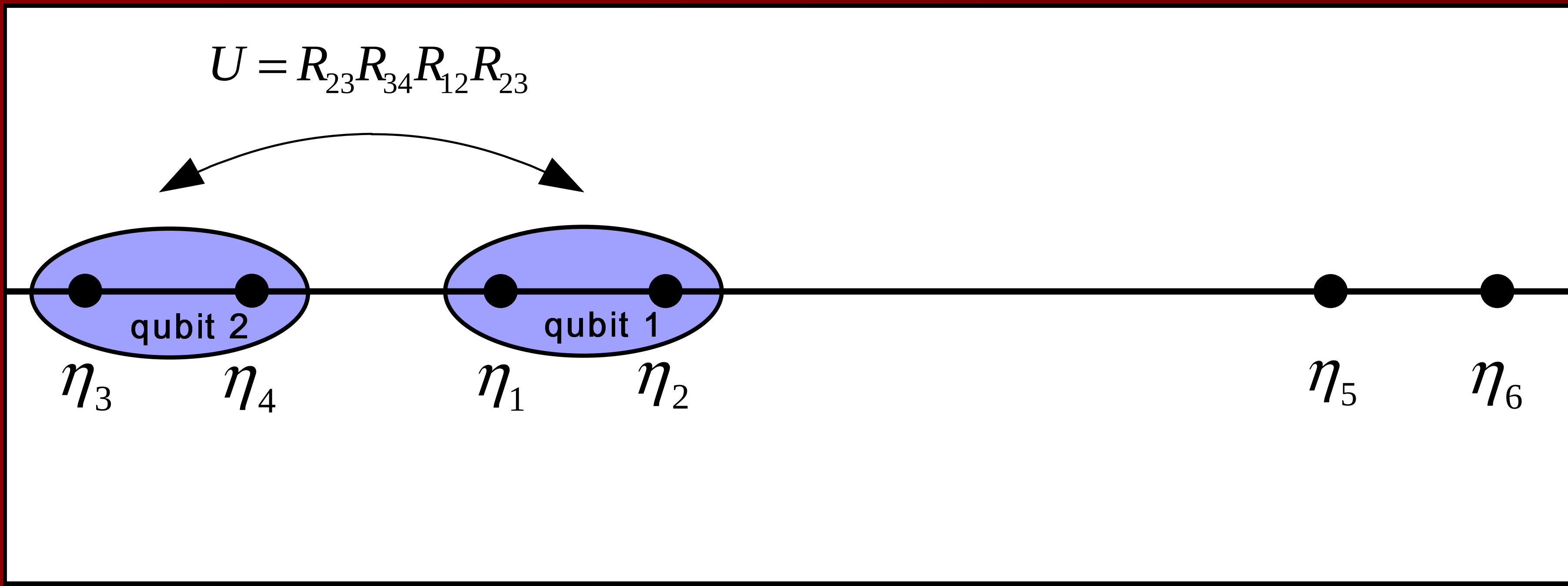

$$U_{(1,2)\leftrightarrow(3,4)} = R_{23}R_{34}R_{12}R_{23} = i\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

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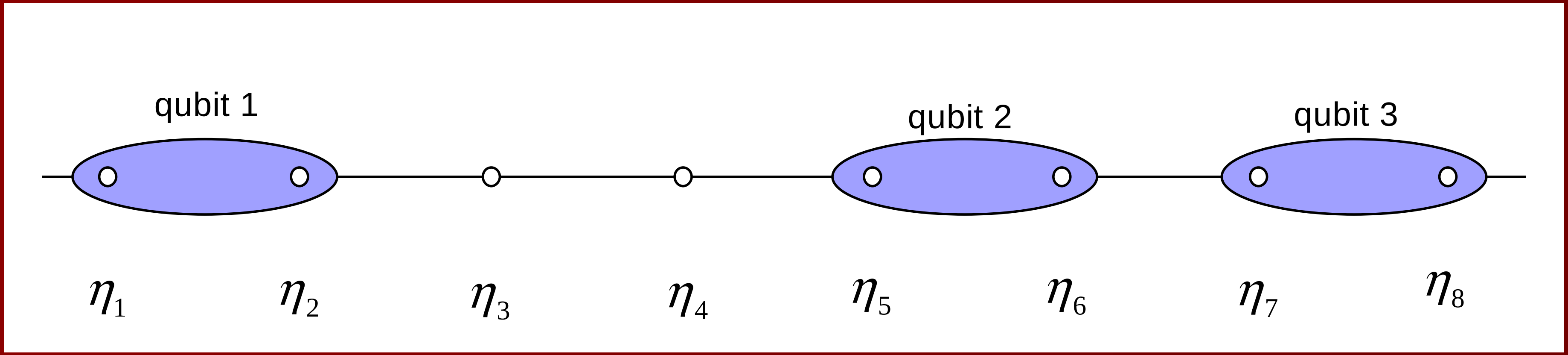
Three qubits and three-qubit gates:



- **Fermion parity conservation:**

$$e_3 e_4 = e_1 e_2 e_5 e_6 e_7 e_8 \qquad \Rightarrow \qquad \#\text{states} = 8.$$

- **Computational basis:**

$$
\begin{aligned}
|000\rangle &\equiv \langle\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\rangle, \\
|001\rangle &\equiv \langle\sigma_+\sigma_+\sigma_+\sigma_-\sigma_+\sigma_+\sigma_+\sigma_-\rangle, \\
|010\rangle &\equiv \langle\sigma_+\sigma_+\sigma_+\sigma_-\sigma_+\sigma_-\sigma_+\sigma_+\rangle, \\
|011\rangle &\equiv \langle\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\sigma_-\sigma_+\sigma_-\rangle, \\
|100\rangle &\equiv \langle\sigma_+\sigma_-\sigma_+\sigma_-\sigma_+\sigma_+\sigma_+\sigma_+\rangle, \\
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|111\rangle &\equiv \langle\sigma_+\sigma_-\sigma_+\sigma_-\sigma_+\sigma_-\sigma_+\sigma_-\rangle.
\end{aligned}
$$

Exchange matrices for 8 quasiholes

- **Recursive: obtain from** $R^{(6)}_{a,a+1}$
- **Fuse** $\eta_7 \to \eta_8$ **or** $\eta_5 \to \eta_6$ **or** $\eta_3 \to \eta_4$ **etc.**

$$R^{(8)}_{12} \;=\; \mathrm{diag}(1,1,1,1,i,i,i,i) = R^{(6)}_{12} \otimes \mathbb{I}_2,$$

$$R_{23}^{(8)} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 1 \end{bmatrix} = R_{23}^{(6)} \otimes \mathbb{I}_2$$

$$R_{34}^{(8)} = \mathrm{diag}(1, i, i, 1, i, 1, 1, i),$$

$$R_{45}^{(8)} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i & 0 & 0 & 0 & 0 \\ -i & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 & -i & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 1 \end{bmatrix} = R_{45}^{(6)} \otimes \mathbb{I}_2$$

$$R_{56}^{(8)} = \mathrm{diag}(1, 1, i, i, 1, 1, i, i) = R_{56}^{(6)} \otimes \mathbb{I}_2,$$

$$R_{67}^{(8)} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 1 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 1 \end{bmatrix},$$

$$R_{78}^{(8)} = \mathrm{diag}(1, i, 1, i, 1, i, 1, i) = \mathbb{I}_2 \otimes R_{56}^{(6)}.$$

- **Artin relations for $\mathcal{B}_8$**

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- **Artin relations for $\mathcal{B}_8$**

$|\mathrm{Image}(\mathcal{B}_8)| = 5160960, \quad |\mathrm{Image}(\mathcal{M}_8)| = 128$

The universal R matrix for $U_q(sl(2))$

- **n-qubit spaces:** $2(n+1)$ Ising anyons realize spinor IR's of $SO(2n+2)$ (of dimension 2^n)

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Spinor representation of the R matrix for the Pfaffian FQH state (Ising model) [Nayak-Wilczek, Slingerland-Bais, Ivanov]:

- **Case k odd:** R acts only on the k-th factor

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- **Case k even:** R acts on the k-th and $(k+1)$-th factor

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{bmatrix}$$

N.B.: Project $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ to the two-qubit basis $\{|000\rangle, |011\rangle, |101\rangle, |110\rangle\}$

$$P_2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Factorizability lost after projection**

$$R_{34}^{(6)} = P_2 \left(\mathbb{I}_2 \otimes \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \otimes \mathbb{I}_2 \right) P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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• **Use universal R matrices for TQC** [Kauffman]

• **Oportunity for topological quantum entanglement**

Embed 1-qubit and 2-qubit gates into 3-qubits

- **Phase gates:**

$$S_1 \equiv S \otimes \mathbb{I}_4 = R^{(8)}_{12}, \quad S_2 \equiv \mathbb{I}_2 \otimes S \otimes \mathbb{I}_2 = R^{(8)}_{56},$$
$$S_3 \equiv \mathbb{I}_4 \otimes S = R^{(8)}_{78}$$

- **Hadamard gates:**

$$
\begin{aligned}
H_1 &= H \otimes \mathbb{I}_4 \simeq R_{12}^{-1} R_{23}^{-1} R_{12}^{-1} \\
&= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$

$$H_2 \simeq \mathbb{I}_2 \otimes H \otimes \mathbb{I}_2 \simeq R_{56}^{-1} R_{45}^{-1} R_{56}^{-1}$$

$$= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} \left[\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{array} \right].$$

Third Hadamard gate reproduced up to swapping

$$\begin{aligned} H_3 &= \mathbb{I}_4 \otimes H \simeq R_{78}^{-1} R_{45}^{-1} R_{56}^{-1} R_{67}^{-1} R_{56}^{-1} R_{45}^{-1} R_{78}^{-1} \\ &= \frac{-\mathrm{e}^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Toffoli gate by Controlled-S and CNOT

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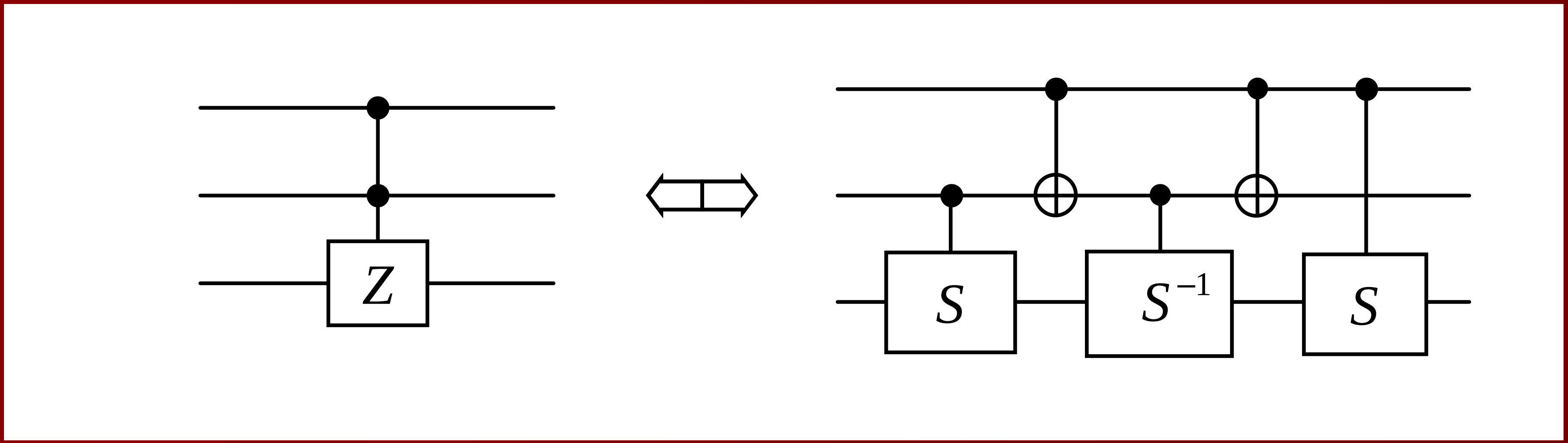
- **Construct CCZ gate bay Controlled-S (CS) and CNOT:**

$$\mathrm{CS} = \mathrm{diag}(1,1,1,i)$$


Z
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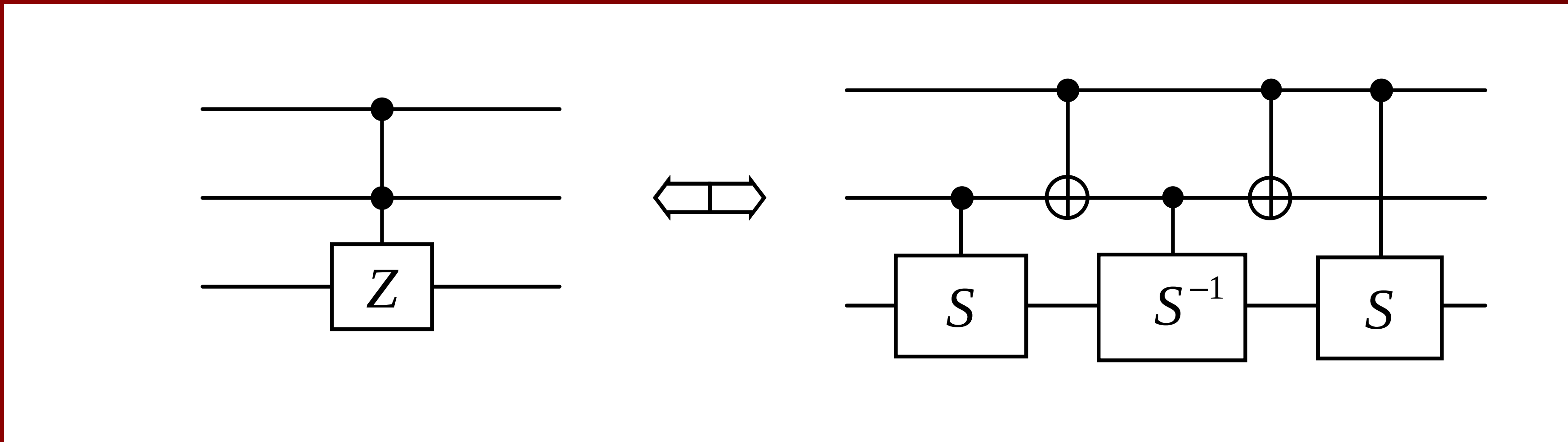
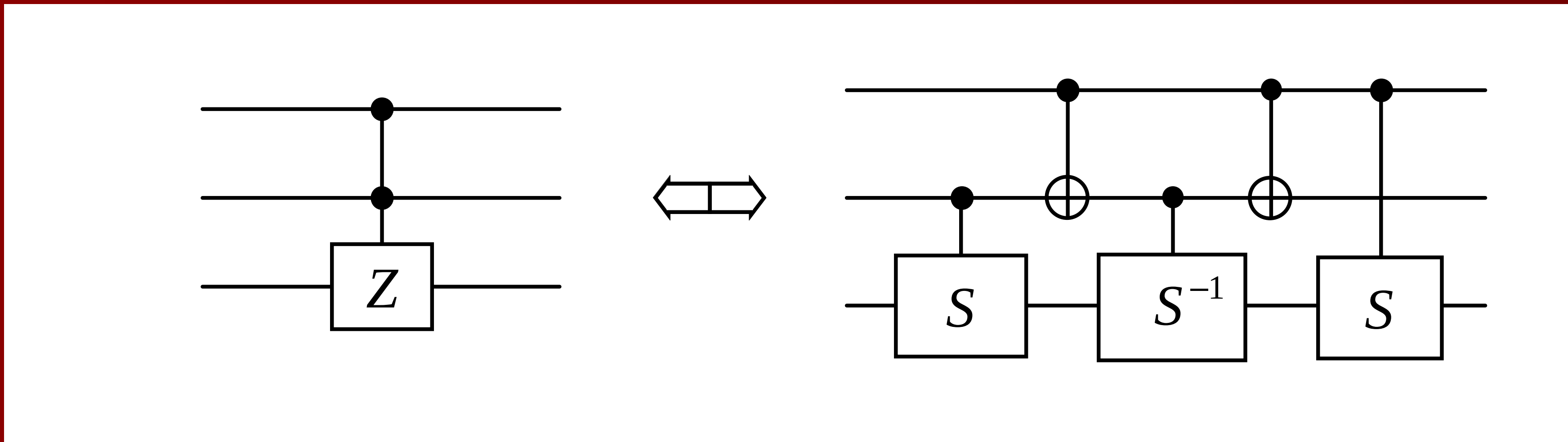
S



- **CS cannot be expressed by braiding:**



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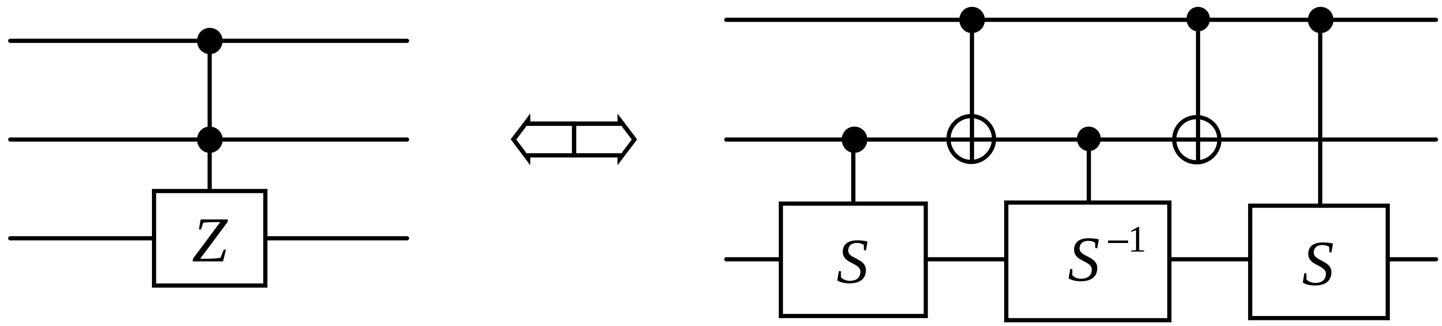

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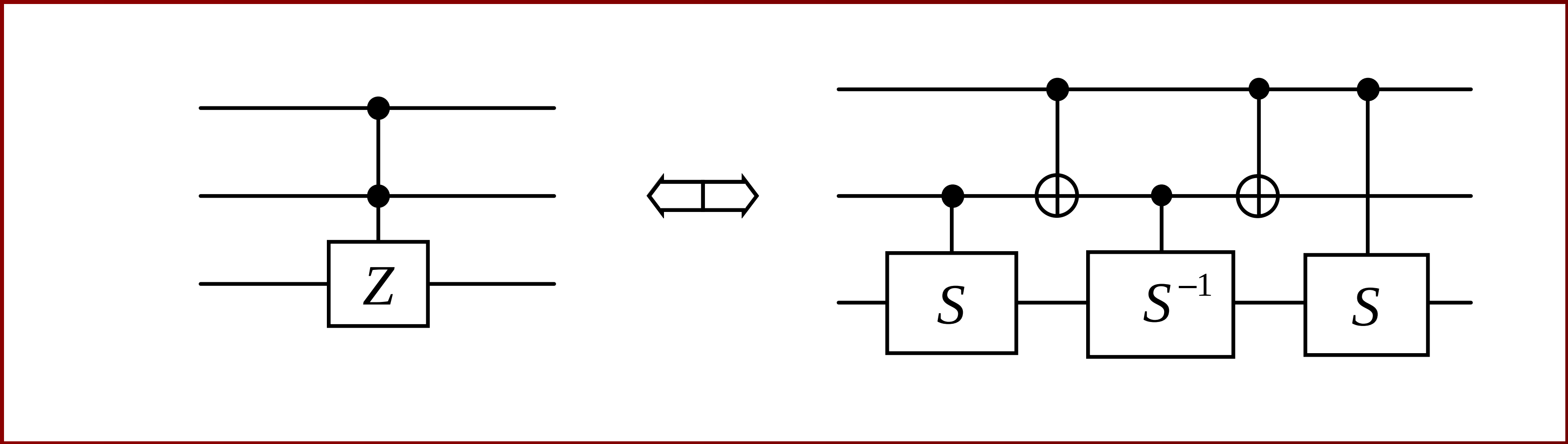

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- **Braid precursor:** $R^{(6)}_{56} = \mathrm{diag}(1, i, 1, i)$

- **Bravyi–Kitaev construction:** split qubit 1 into two charge-$1/4$ quasiholes only if it is in the state $|1\rangle$

Toffoli gate by braid-group CCZ precursor

$$\widetilde{\mathrm{CCZ}} \equiv R_{12}^{(8)} R_{34}^{(8)} R_{56}^{(8)} R_{78}^{(8)} \simeq \mathrm{diag}\left(-1, 1, 1, 1, 1, 1, 1, -1\right)$$

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- **Dehn twists!**

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- Analyze the error sources