Entanglement entropy at quantum critical points: Can you hear the shape of Schrödinger's cat?

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Outline

- Motivation
- Entanglement entropy
- Entanglement entropy and quantum criticality in $1 + 1$ Dimensions
- Topological Phases, Topological Field Theories and Quantum Phase Transitions
- Conformal Quantum Critical Points in $2 + 1$ dimensions
- Entanglement Entropy of Scale Invariant Wave Functions

Motivation

- Entanglement entropy is a clear measure of quantum mechanical correlations in many body systems and field theories.
- It is however a very non-local quantity which very difficult to measure in macroscopic quantum systems.
- Its behavior near quantum critical points is largely not understood except in one dimension.
- Recent results show it is a quantity of interest in topological phases.

Entanglement Entropy in Quantum Mechanics

- Consider a quantum mechanical system with two sets of degrees of freedom, $\{\varphi_A\}$ and $\{\varphi_B\}$. Given the pure state $\Psi[\varphi_A, \varphi_B]$ and the trivial density matrix for the combined system $A \cup B$,

$$\langle \varphi_A, \varphi_B | \rho_{A\cup B} | \varphi'_A, \varphi'_B \rangle = \Psi[\varphi_A, \varphi_B] \Psi^* [\varphi'_A, \varphi'_B]$$

- The reduced density matrix for A is constructed by tracing over the degrees of freedom in B:

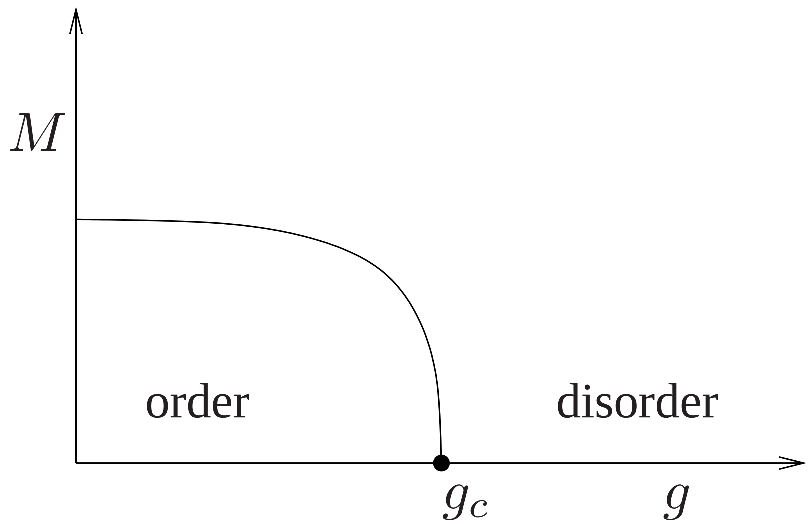
$$\langle \varphi_A | \rho_A | \varphi'_A \rangle = \mathrm{tr}_B \ \rho_{A\cup B}$$

- The von Neumann entanglement entropy is

$$S_A = -\mathrm{tr}_A \ (\rho_A \ln \rho_A) = -\mathrm{tr}_B \, (\rho_B \ln \rho_B) = S_B$$

- The von Neumann entropy gives a measurement of how entangled (or correlated) the degrees of freedom of A are with those of B.

Quantum Phase Transitions and Quantum Criticality



- Phase transitions at zero temperature tuned by a parameter g of a quantum Hamiltonian
- Scale Invariance at Quantum Critical Points: Correlation functions of local operators exhibit scaling.
- As the transition there is a diverging correlation length $\xi \sim |g - g_c|^{-\nu}$
- The energy gap Δ closes with a power law $\Delta \sim |g - g_c|^{\nu z}$, where z is the dynamic critical exponent
- For systems which exhibit an effective Lorentz invariance at criticality $z = 1$; in general $z \neq 1$.

$D = 1$ Quantum Criticality: Conformal Invariance

- Quantum critical systems with $z = 1$ in one dimension are special
- In their $1 + 1$-dimensional space-time they exhibit an infinite conformal invariance: Conformal Field Theory
- The critical exponents are quantum numbers that label the representations of the (infinite-dimensional) group of conformal transformations
- (Almost) Complete classification of quantum phase transitions of unitary relativistic ($z = 1$) systems in $1 + 1$ dimensions in terms of Conformal Field Theory (CFT)
- The central charge c (or conformal anomaly) and the coefficients of the operator product expansion (OPE) of the primary fields define the universal properties of the CFT.
- Universal form of the low temperature specific heat C/L and of the ground state energy density E/L

$$\frac{C}{L} = \frac{\pi}{6} \frac{cT}{v}, \qquad \frac{E}{L} = \varepsilon_0 + \frac{\pi}{6} \frac{cv}{L}$$

Entanglement and Conformal Invariance

- Calculations on a number of spin chains led to the suggestion that the von Neumann entropy in $D = 1$ quantum critical systems has a universal term which scales as $\log L$ (Rico, Latorre, Vidal, and Kitaev)

- Calbrese and Cardy (2004) (and Holzhey, Larson and Wilczek (1994)) used conformal invariance to show that, for a large region of linear size L, embedded in an infinitely long $1+1$-dimensional conformally invariant system, the von Neumann entanglement entropy has the asymptotic behavior

$$S = \frac{\pi}{3} c \, \log \left(\frac{L}{a} \right) + \text{finite terms}$$

where s_0 is non-universal, a is a short-distance cutoff, and c is the central charge

- Away from criticality: there is a finite correlation length ξ.

$$S = \frac{\pi}{3} c \, \log \left(\frac{\xi}{a} \right) + \text{finite terms}, \qquad \text{for } \xi \gg a$$

- $1+1$ dimensional disordered quantum systems at *random fixed points* also obey a logarithmic scaling law (even though they are scale but not conformal invariant!) (Refael and Moore, 2004).

Issues on Scaling Behavior of Quantum Entanglement

- Behavior of quantum entanglement near quantum critical points in general and in higher dimensions.
- Dependence on the dynamic quantum critical exponent z which determines the nature of quantum criticality for local operators.
- Massive relativistic ($z = 1$) (free) field theories obey an "area law" $S = AL^{d-1}$ (Srednicki, 1993) (where A is non-universal).
- Calabrese and Cardy (2004) have used a heuristic scaling argument to argue that the area law is generic to quantum critical points in all dimensions.
- We will see however that, for conformal quantum critical points in $2+1$-dimensions (with $z = 2$) there is in fact a universal logarithmic term.
- Universal *finite* terms appear in *topological phases* and depend on the quantum dimensions.

Spin Liquids and Topological States of Matter

- Liquid phases of electron fluids and spin systems without long range order, with or without time reversal symmetry breaking
- Quasiparticles: vortices with fractional charge and fractional statistics (Abelian and non-Abelian)
- Hidden Topological Order and Topological Vacuum Degeneracy
- Finite-dimensional quasiparticle Hilbert spaces $\Rightarrow$ universal topological quantum computer

"Known" Topological Quantum Liquids

- 2DEG Fractional Quantum Hall Liquids
  - Abelian FQH states (Laughlin and Jain): fractional charge and Abelian fractional statistics
  - Non-Abelian FQH states: Is $\nu = 5/2$ a Pfaffian (Moore-Read) FQH state? (firm candidate) Is the plateau at $\nu = 12/5$ a parafermion state? (good possibility)
- Rapidly rotating Bose gases: possible non-Abelian (Pfaffian) FQH state of bosons at $\nu = 1$
- Time-Reversal Breaking Superconductors: Sr_2RuO_4 is a $p_x + ip_y$ superconductor

Challenges

- To develop a consistent theory of topological phases (i.e. beyond FQH states) and to understand the underlying mechanisms
- What are the generic phases of models of topological liquids
- Is the gap necessary? Can a topological liquid be gapless?
- Concrete examples of lattice models with local interactions with topological phases
- Fractional Statistics: Abelian and non-Abelian
- There has been some progress in constructing models with Abelian statistics
- To find experimentally realizable models (looks promising, not quite there yet)

Time Reversal Invariant Spin Liquids: Quantum Dimer Models

- Simple local models describing strongly frustrated and ring exchange quantum spin systems with a large spin gap and no long range spin order
- They typically exhibit spin gap phases with different types of valence bond crystal orders
- QDM have special solvable points, the Rokhsar-Kivelson (RK) point, where the exact ground state wave function has the short range RVB form

$$|\Psi_{\mathrm{RVB}}\rangle = \sum_{\{C\}} |C\rangle, \qquad \{C\} = \text{ all dimer coverings of the lattice}$$

- – Bipartite lattices: the RK points are quantum (multi) critical points, described by an effective field theory with $z = 2$ and massless deconfined spinons, or first order transitions
  – Non-bipartite lattices: QDMs have topological $\mathbb{Z}_2$ deconfined phases with massive spinons and a topological 4-fold ground state degeneracy on a torus (Moessner and Sondhi, 1998)

The Quantum Dimer Model

$$H_{\mathrm{RK}} = \sum_i (vV_i - tF_i), \qquad \text{Rokhsar and Kivelson (1988)}$$

Here each bar represents a spin singlet bond.

For $t = v$ $\quad \Rightarrow H_{\mathrm{RK}} = \sum_i Q_i^\dagger Q_i$, with $Q_i = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$.

- The ground state wave function $|\Psi_0\rangle$ has $E = 0$

$$|\Psi_0\rangle = \frac{1}{\sqrt{Z_{\mathrm{cl}}}} \sum_C |C\rangle \ ,$$

  where Z_{cl} is the sum over all dimer configurations

- Equal-*time* correlators in the quantum dimer model at the RK point are given by correlators of the classical dimer model.

- This is actually a loop model: loops are the dimer moves from a reference state. This is the simplest loop model: the $SU(2)_1$ fully packed loop model.

Dimers, heights and effective field theory

Moessner, Sondhi and Fradkin; Ardonne, Fendley and Fradkin

- The QDM can be mapped to a height model
- Plaquette flip changes the height of that plaquette by ± 4, and the average height of the surrounding sites by ± 1.
- Equivalent configurations: $h \cong h + 4$.
- Continuum limit: $h \cong 4\varphi(x)$
  Compactification Radius: $\varphi(x) \cong \varphi(x) + 1$.
- The Quantum Lifshitz model
  Hamiltonian:
  $$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{\kappa^2}{2} \left(\nabla^2 \varphi\right)^2 \right]$$
  This is the Quantum Lifshitz Model. (Henley; Moessner, Sondhi and Fradkin)
- Action in imaginary time $\tau \Leftrightarrow$ smectic layers in 3D classical statistical mechanics at the Lifshitz transition.
  $$S = \int d^2x \int d\tau \left[\frac{1}{2} \left(\partial_\tau \varphi\right)^2 + \frac{\kappa^2}{2} \left(\nabla^2 \varphi\right)^2 \right]$$

Scale Invariant Ground State Wave Functions and 2D Classical Critical Phenomena

- $$\int d^2\vec{x}\left[-\frac{1}{2}\left(\frac{\delta}{\delta\varphi}\right)^2 + \frac{\kappa^2}{2}(\nabla^2\varphi)^2\right]\Psi[\varphi] = E\Psi[\varphi]$$

$$Q(\mathbf{x}) \equiv \frac{1}{\sqrt{2}}\left(\frac{\delta}{\delta\varphi} + \kappa\nabla^2\varphi\right) \qquad Q^\dagger(\mathbf{x}) \equiv \frac{1}{\sqrt{2}}\left(-\frac{\delta}{\delta\varphi} + \kappa\nabla^2\varphi\right)$$

- Ground state wave-function, $\Psi_0[\varphi]$

$$Q(\vec{x})\Psi_0[\varphi] = 0 \qquad \Rightarrow \qquad \Psi_0[\varphi] \propto e^{-\frac{\kappa}{2}\int d^2x\ (\nabla\varphi(\mathbf{x}))^2}$$

$$\|\Psi_0\|^2 = \int \mathcal{D}\varphi\ e^{-\kappa\int d^2x\ (\nabla\varphi(\mathbf{x}))^2}$$

- The ground state wave function is conformally invariant

Mapping to a 2D $c = 1$ Euclidean CFT

- The probability for a configuration $|\varphi\rangle$ is the Gibbs weight of a 2D classical Gaussian model, a Euclidean 2D free massless scalar field.

- At these quantum critical points the ground state wave function is scale invariant

- The equal-time expectation values of the observables are correlators in this $c = 1$ conformal field theory.

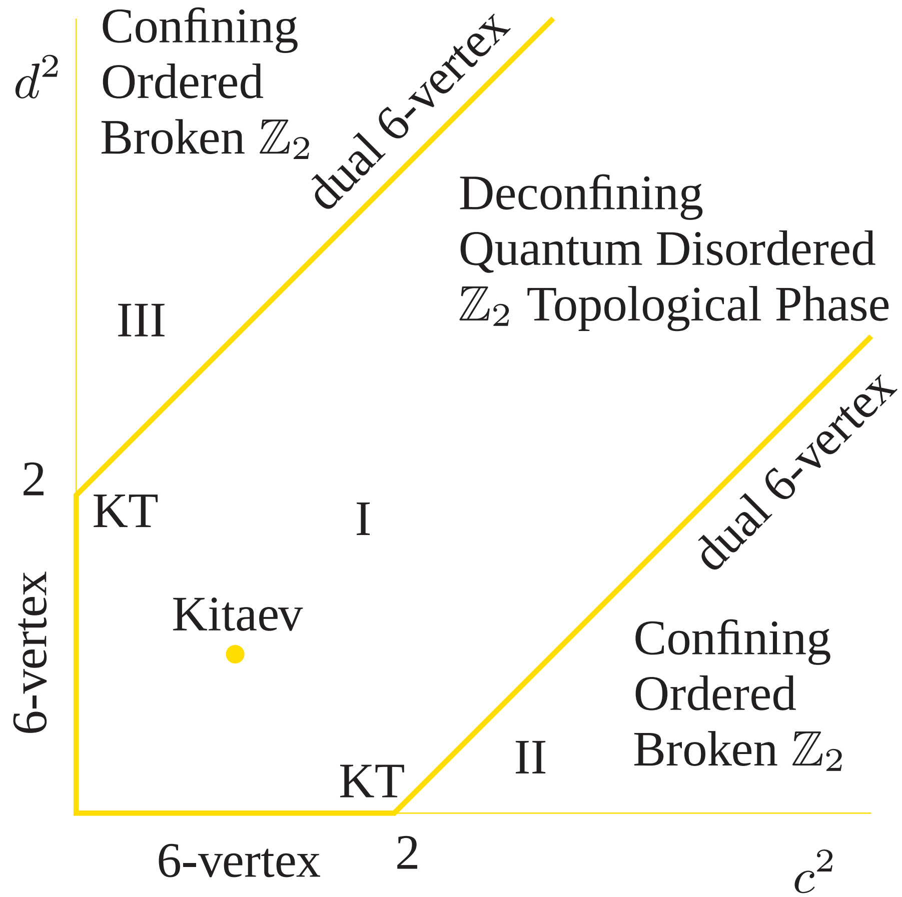
- The equal-time expectation value for operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$. Time-dependent correlators exhibit power-law behavior with dynamical exponent $z = 2$.

- Matching the correlation functions of the RK and Lifshitz models, one finds $\kappa = 1/2\pi$.

- This is a multicritical point with many relevant perturbations: e.g diagonal dimers drive the system into a $\mathbb{Z}_2$ topological phase

Phase diagram for a quantum eight vertex model

Ardonne, Fendley and Fradkin



Strategy for a Generalization

with Paul Fendley

- Each basis state in the Hilbert space is a loop configuration in 2D
- Start with the statistics we wish to have, and work backward
- Algebraic characterization of braiding for $SU(2)_k$ and $SO(3)_k$ Chern-Simons
- Braid matrix of a 2+1-dimensional theory as a limit of the S-matrix of an associated relativistic 1+1 dimensional model
- We construct quantum 2D models with these braid relations by utilizing the structure of the factorizable S-matrices of integrable 1D models.
- We embed the 1D model in 2D Euclidean space, and find an RK Hamiltonian
- Loop gases:
  - $SU(2)_k$ case: $O(n)$ loop lattice model with $n = 2\cos(\pi/(k+2))$ (self-avoiding and mutually-avoiding loops)
  - $SO(3)_k$ case: domain walls of a Q-state Potts model with $Q = 4\cos^2(\pi/(k+2))$ (loops intersect and branch: nets)

    $SO(3)_3$: Chromatic Polynomial "Loop" gas wave function and Fibonacci anyons.

Entanglement Entropy of 2D Quantum Critical States

with Joel Moore

- Kitaev and Preskill (2006), and Levin and Wen (2006) showed that the entanglement (von Neumann) entropy S of a region of linear size L in 2D topological phases has the behavior

$$S = \alpha L - \gamma + O(1/L)$$

  α is a non-universal coefficient and $\gamma = \ln \mathcal{D}$ is a finite universal constant, $\mathcal{D} = \sqrt{\sum_i d_i^2}$, determined by the quantum dimensions d_i of the excitations of the topological phase.

- This topological entropy plays a crucial role in single point contacts in non-Abelian FQH states (Fendley, Fisher and Nayak, 2006) and gives new meaning to the boundary entropy of quantum impurity problems and 1D boundary CFTs (Affleck and Ludwig)

- The proximity of the 2D conformal quantum critical points we discussed here to 2D topological phases suggest that they may hold clues on this behavior.

- Is there a universal signature in the von Neumann entropy of quantum critical systems?

Can you hear the shape of Schrödinger's Cat?

- For conformal quantum critical points, the Hilbert space has an orthonormal basis of states $|\{\phi\}\rangle$ indexed by classical configurations $\{\phi\}$, and the ground state $|\Psi_0\rangle$ of the bipartite system is determined by a CFT action S:

$$|\Psi_0\rangle = \frac{1}{\sqrt{Z_c}} \int (d\phi)\, e^{-S(\{\phi\})/2} |\{\phi\}\rangle.$$

$Z_c = \int (d\phi)\, e^{-S(\{\phi\})}$, and expectation values are CFT correlators.

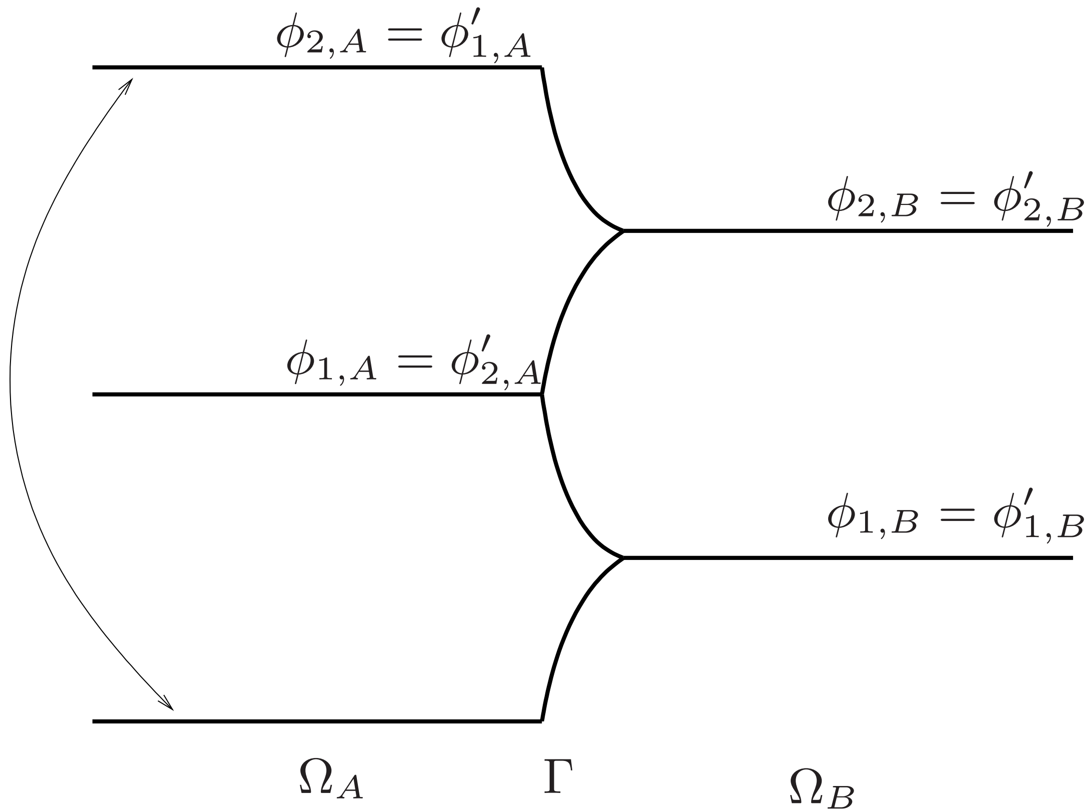
- Entanglement Entropy

$$S = -\mathrm{Tr}\, \rho_A \log \rho_A = - \frac{\partial \mathrm{Tr}\, \rho_A^n}{\partial n}\Big|_{n=1}.$$

- $$\begin{aligned}&\langle\{\phi_1^A\}|\rho_A|\{\phi_2^A\}\rangle = \mathrm{Tr}_{\phi^B}\left(\langle\{\phi_1^A\}|\otimes\langle\{\phi^B\}|\psi_0\rangle\langle\psi_0(|\{\phi_2^A\}\rangle\otimes|\{\phi^B\}\rangle\right.\\ &= \frac{1}{Z_c}\int(d\phi^B)e^{-(S^A(\phi_1^A)/2+S^A(\phi_2^A)/2+S^\partial(\phi_1^A,\phi^B)/2+S^\partial(\phi_2^A,\phi^B)/2+S^B(\phi^B))}.\end{aligned}$$

  Here the action has been divided into regions A, B, and the boundary ∂, where the last takes into account contributions mixing the A and B degrees of freedom

- Higher powers of the density matrix need not trace to unity: $\mathrm{Tr}\,\rho_A^n$ is now a sum over n configurations defined in A and n configurations defined in B. The key is to keep track of how these different configurations are stitched together at the boundary by the terms S^∂ that link A and B: $\{\phi_i^A\}$ is linked to $\{\phi_i^B\}$ as well as $\{\phi_{i+1}^B\}$ for $i = 1, \ldots, n-1$, and $\{\phi_n^A\}$ is linked to $\{\phi_n^B\}$ and $\{\phi_1^B\}$. This is normalized through division by $(Z_c)^n$, which can be thought of again as n copies of A and B configurations, but with $\{\phi_i^A\}$ linked only to $\{\phi_i^B\}$.


$\phi_{2,A} = \phi'_{1,A}$
$\phi_{2,B} = \phi'_{2,B}$
$\phi_{1,A} = \phi'_{2,A}$
$\phi_{1,B} = \phi'_{1,B}$
Ω_A
Γ
Ω_B

- The boundary terms impose continuity of the fields in the CFT because strong local fluctuations are penalized in the action. A key feature of the replicas in the numerator is that each replica on A is connected symmetrically to two different replicas on B and vice versa. The symmetry, which for N replicas is the Schoenflies group C_{Nv} (the cyclic group plus vertical mirror plane), prevents a unique idenfitication of replicas on A with replicas on B except for $N = 1$, so the numerator cannot be equivalent to the denominator (N free fields on $A \cup B$). Instead the fields from different replicas are forced to agree on the boundary: schematically

$$\mathrm{Tr}\,\rho_A^n = \frac{Z(n \text{ configurations agreeing on the boundary})}{Z(n \text{ independent configurations})} = \mathrm{Tr}\,\rho_B^n$$

- $\mathrm{Tr}\,\rho_A^n$ can be put in a form that simplifies taking the derivative as $n \to 1$: for an explicit realization, consider the case of a free scalar field. Then the condition that n scalar fields ϕ_i agree with each other on the boundary can be satisfied by forming $n - 1$ linear combinations $\frac{1}{\sqrt{2}}(\phi_i - \phi_{i+1})$, which vanish at the boundary i.e., satisfy Dirichlet boundary conditions, plus one linear combination $\frac{1}{\sqrt{n}} \sum_{i=1,\ldots,n} \phi_i$ that has no boundary condition (i.e., is a free field on $A \cup B$).

- For any CFT there exists a conformal boundary condition that generalizes the notion of the Dirichlet boundary condition in the free case. In terms of the partition functions Z_D, for a field in the whole system $A \cup B$ that vanishes at the boundary, and Z_F, for a field that is free at the boundary,

$$\mathrm{Tr}\,\rho_A^n = \frac{Z_D^{n-1} Z_F}{{Z_F}^n} = \left(\frac{Z_D}{Z_F}\right)^{n-1}$$

$$S = -\log \frac{Z_D}{Z_F} = -\log \frac{Z_D^A Z_D^B}{Z_F}.$$

- In the last equality, the Dirichlet boundary condition at the boundary was used to split the partition function into contributions from A and B, each including the boundary with Dirichlet boundary conditions.

- The entanglement entropy for a general conformal quantum critical point is just the dimensionless free energy difference induced by the partition in the associated CFT:

$$S = F_A + F_B - F_{A \cup B}.$$

- For the QCPs we are discussing here Z is the partition function of a CFT with Dirichlet BCs. For a bounded region of linear size L and smooth boundary, it obeys the 'Mark Kac law' ('Can you hear the shape of a drum?')

$$F = aL^2 + bL - \frac{c}{6}\chi \log L + O(1) \qquad \text{(Cardy and Peschel)}$$

where a and b are non-universal, and χ is the Euler characteristic of the region (manifold):

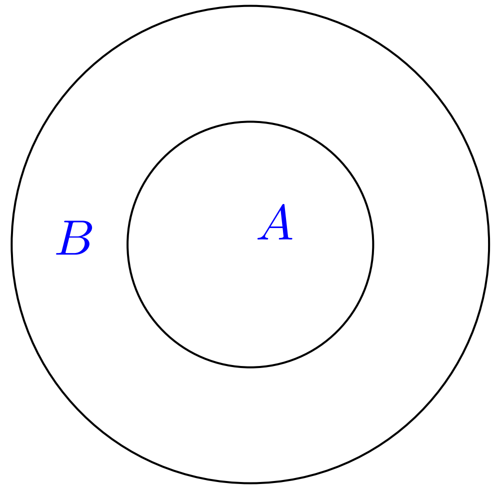
$$\chi = 2 - 2h - b, \qquad h = \#\ \text{handles},\ b = \#\ \text{boundaries}$$

- This result suggests the existence of a $\log L$ dependence with an universal coefficient associated with the central charge c of the associated CFT

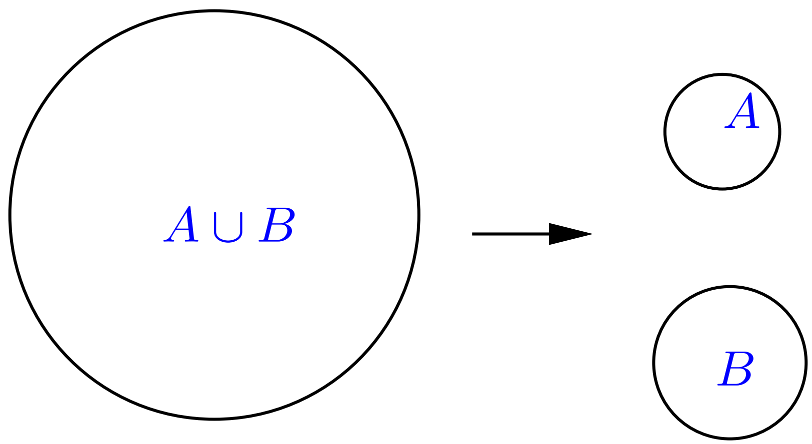
- This result implies that for a QCP described by a scale (and conformally) invariant ground state wave function, the entanglement entropy of regions A and B with a smooth common boundary has a universal logarithmic term of the form

$$\Delta S = -\frac{c}{6}\left(\chi_A + \chi_B - \chi_{A\cup B}\right) \log L$$

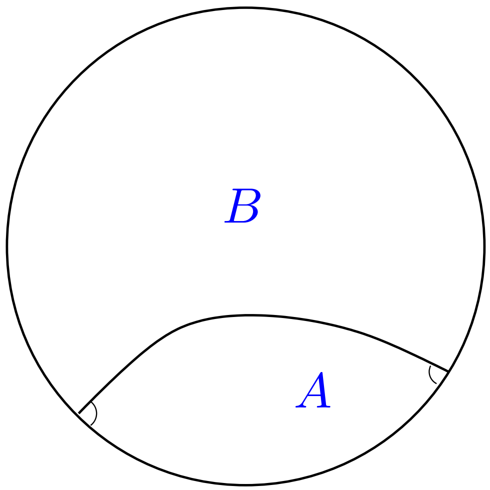
For regions $A \subseteq B$ the coefficient of the $\log L$ is zero since in this case $\chi_A + \chi_B = \chi_{A\cup B} \Rightarrow \Delta S = 0$



If the regions A and B are physically separate and have no common intersection, $\chi_A + \chi_B - \chi_{A\cup B} \neq 0$. In this case, which corresponds to a process in which the system physically splits in two disjoint parts, there is a universal $\log L$ term in the entanglement entropy at quantum criticality, proportional to the central charge c of the associated CFT!

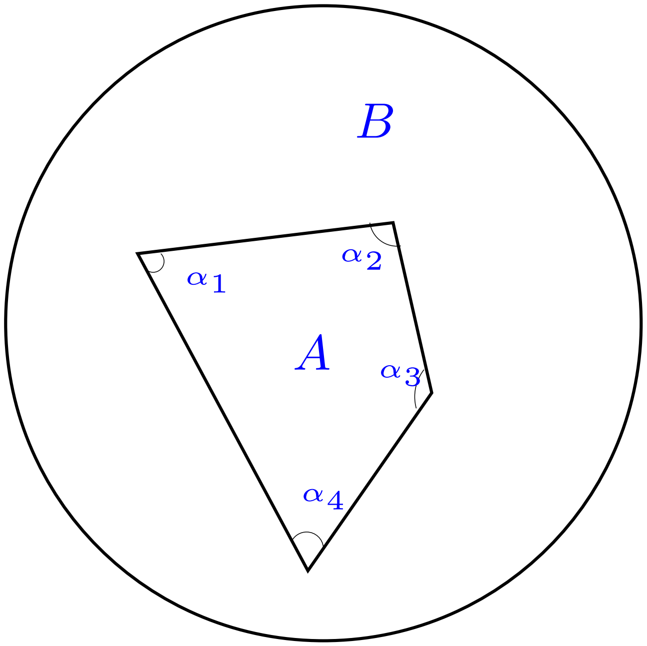


If the A and B share a common boundary, there is a $\log L$ term whose coefficient is determined by the angles at the intersections



Or if the boundary of A is not smooth, in which case the coefficient depends on the angles α_i for both regions

$$(\Delta S)_i = \frac{c\,\alpha_i}{24\pi}\left(1-\left(\frac{\pi}{\alpha_i}\right)^2\right)\,\log L$$



Finite terms in the entanglement entropy depend on scale-invariant aspect ratios

Away from Quantum Criticality: Topological Phases

- Topological Phases are proximate to conformal quantum critical points and can be accessed by relevant perturbations

- Topological Phases have a finite correlation length ξ.

$$S = \frac{c'}{3} \log \frac{\xi}{a} + \ldots \quad \text{for } \xi \gg a$$

- For $a < \xi < \infty$, the entropy is defined by a crossover scaling function whose behavior is controlled by the correlation length ξ

- At the stable fixed point, $\xi \to a$

$$S = s_0 \frac{L}{a} - \gamma$$

γ is universal (Kitaev and Preskill (2006); Levin and Wen (2006).)

Conclusions

- We examined the behavior of the Von Neumann entanglement entropy for a class of quantum critical points in $2+1$ dimensions with conformally invariant *wave functions*
- These quantum critical points are *proximate* to *topological phases* and can be used to access them
- The entanglement entropy at these conformal quantum critical points also has universal logarithmic terms which become manifest in *topology changing processes*
- This result suggests that the entanglement entropy is sensitive to global properties not only in topological phases but also at quantum criticality.