Majorana Fermions in a Superconductor

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Model : Superconducting Proximity effect at the surface of a Topological Insulator

L. Fu and C. Kane (2008)
R. Jackiw and P. Rossi (1981)

A Dirac type matrix equation governs surface excitations of a topological insulator in contact with an s-wave superconductor.

A vortex configuration in the superconductor leads to a static, isolated **zero energy** solution. Its mode function is real and has been called **Majorana**.

- will show that the Majorana feature is not confined to the zero energy mode, but characterizes the full quantum theory.

- will discuss the quantization procedure examining the Fock space realization of the zero mode algebra for the Dirac-type systems.

Majorana Fermions

Central to recent research in
particle physics - neutrino physics, supersymmetry
cosmology - dark matter
condensed matter physics - exotic superconducting states

What is Majorana fermion ?
electrically charged particles – particle is different from its anti-particle which has opposite charge

electrically neutral particles – particle can be its own anti-particle

examples: neutral pions ($S = 0$), photons ($S = 1$), gravitons ($S = 2$)
they are all bosons! – they are created by fields that obey
$\Phi = \Phi^*$ (reality condition)

fermions $\left(S = \frac{1}{2}\right)$

Dirac equation – complex numbers seem unavoidable
successful theory for understanding
spin and prediction of anti-matter

$$(i\gamma^\mu \partial_\mu - \mathrm{m})\ \Phi = 0 \qquad \Phi : \text{four-component spinor}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \text{(Clifford algebra)}$$

$$\gamma^{0^\dagger} = \gamma^0;\ \gamma^{i^\dagger} = -\gamma^i \quad \text{(Hermitian Hamiltonian)}$$

Majorana's work (1937)

question:	are equations for spin $\frac{1}{2}$ fields necessarily complex?
answer:	there is a simple modification of Dirac equation that involve only real numbers.
$\Rightarrow$	profound implication that spin $\frac{1}{2}$ particles can be its own anti-particles!

Majorana field and its equation of motion

$$(i\tilde{\gamma}^\mu \partial_\mu - m)\ \Psi = 0 \quad \Psi: \text{ four-component spinor}$$

$$\tilde{\gamma}^\mu: \text{ purely imaginary, satisfying Clifford algebra}$$

$\Rightarrow$ Ψ **can** be real! $\Psi = \Psi^*$ reality condition

N.B. purely imaginary $\tilde{\gamma}^\mu$: **Majorana representation** of γ-matrices

To describe Majorana fermions one does not need to use purely imaginary $\tilde{\gamma}^\mu$, but impose

$$\Psi^C \equiv C\Psi^* = \Psi \qquad \text{pseudo-reality condition}$$

where C is charge conjugation matrix.

$$C = \begin{pmatrix} 0 & -i\,\sigma^2 \\ i\,\sigma^2 & 0 \end{pmatrix}$$

Consider a four component spinor $\Phi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$

imposing $\quad C\Psi^* = \Psi \quad \Rightarrow \quad \varphi = i\sigma^2\psi^* \quad (\psi = -i\sigma^2\varphi^*)$

$$\Psi = \begin{pmatrix} \psi \\ i\sigma^2\,\psi^* \end{pmatrix} \qquad \text{two component theory}$$

Dirac: $$\mathcal{L}_D = \bar{\Phi}\, i\, \gamma^\mu\, \partial_\mu \Phi - \tfrac{1}{2}\ \mathrm{m}\, \bar{\Phi}\Phi$$

$$\Rightarrow \qquad (i\, \gamma^\mu\, \partial_\mu - \mathrm{m})\ \Phi = 0$$

Majorana: imposing $C\, \Psi^* = \Psi$

$$\mathcal{L}_M = \psi^{*T}\, i\ (\partial_t - \sigma\cdot\partial)\ \psi - \frac{1}{2}\ \mathrm{m}\ (\psi^{*T}\, i\sigma^2\, \psi^* + h.c.)$$

$$\Rightarrow \qquad i\ (\partial_t - \sigma\cdot\partial)\ \psi - \mathrm{m}\ i\sigma^2\, \psi^* = 0$$

N.B. Compare the mass terms:

Dirac: $\mathrm{m}\bar{\Phi}\Phi$ – preserves all quantum numbers

Majorana : $\mathrm{m}\ (\psi^{*T}\, \sigma^2\, \psi^* + h.c.)$

– does not conserve any quantum numbers!

$\Rightarrow$ No distinction between particle and anti-particle since there are no conserved quantum numbers to tell them apart; particle is its own anti-particle

Are there Majorana fermions in nature?

neutrinos?

– recent development in neutrino physics

experimental observation of neutrino oscillations $\Rightarrow$

- neutrinos have mass (< 0.1eV)
- lepton number is not conserved separately!

$\Rightarrow$ they could be Majorana fermions

Hypothetical Majorana fermions:

- supersymmetry – supersymmetric partners of photon, neutral Higgs boson, etc. are necessarily Majorana fermions
- cosmology – dark matter candidates

Majorana fermions in superconductor in contact with a topological insulator

Superconductor

proximity effects $\Rightarrow$ Cooper pairs

tunnel through to the surface of TI

topological insulator

Hamiltonian density for the model:

$$H = \psi^{*T}\ (i\,\sigma \cdot \partial - \mu)\ \psi + \frac{1}{2}\ (\triangle \psi^{*T}\, i\,\sigma^2\, \psi^* + h.c.)$$

$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \sigma = (\sigma^1, \sigma^2), \mu$ is chemical potential and $\triangle$ is the order parameter that may be constant or takes vortex profile, $\triangle(r) = v(r)e^{i\theta}$.

Equation of motion: $i\,\partial_t\,\psi = (\sigma \cdot \partial - \mu)\ \psi + \triangle\, i\,\sigma^2\,\psi^*$

In the absence of μ, and $\triangle$ constant, the above system is a (2+1)-dimensional version of (3+1)-dimensional, two component Majorana equation! − governs chargeless spin $\frac{1}{2}$ fermions with Majorana mass $|\triangle|$.

Quantum Structure of the model / Quantization:

ψ mixes with its complex conjugates in the equation of motion.

$\Rightarrow$ cannot construct energy eigenvalue problem

quantization is carried out both in particle physics and in superconductor by promoting the two component description to a constrained four component description:

$$\Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \\ \psi_\downarrow^* \\ -\psi_\uparrow^* \end{pmatrix} = \begin{pmatrix} \psi \\ i\sigma^2 \psi^* \end{pmatrix} .$$

An extended Hamiltonian density $\mathcal{H}$ leads to equations for ψ, which are just two copies of $\mathcal{H}$:

$$\mathcal{H} = \frac{1}{2} \Psi^{*T} \begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{p} - \mu & , & \triangle \\ \triangle^* & , & -\boldsymbol{\sigma} \cdot \boldsymbol{p} + \mu \end{pmatrix} \Psi \equiv \frac{1}{2} \Psi^{*T} h \, \Psi$$

Bogoliubov-de Gennes equations for superconductor:

Solve energy eigenvalue problem for unconstrained four component spinor $\Phi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$

$$h\Phi = i\,\partial_t\,\Phi \;,\; \Phi = e^{-i\,E\,t}\,\Phi_E$$
$$h\Phi_E = E\,\Phi_E \;.$$

$\Rightarrow$ construct Dirac field operator

$$\begin{aligned} \hat{\Phi} &= \sum_{E>0} a_E\; e^{-i\,E\,t}\;\Phi_E \;+\; \sum_{E<0} b^\dagger_{-E}\,e^{-i\,E\,t}\,\Phi_E \\ &= \sum_{E>0} \left\{ a_E\,e^{-i\,E\,t}\,\Phi_E \;+\; b^\dagger_E\,e^{i\,E\,t}\,C\Phi^*_E \right\} \end{aligned}$$

charge conjugation symmetry of h has been used, $C\,\Phi^*_{+E} = \Phi_{-E}$

Quantum field for superconductor:

$\widehat{\Phi} \to \widehat{\Psi}$ satisfying constraint $C\widehat{\Psi}^\dagger = \widehat{\Psi}$

$$\widehat{\Psi} = \sum_{E>0} \left(a_E \, e^{-i\,E\,t} \, \Phi_E + a_E^\dagger \; e^{i\,E\,t} \, C\Phi_E^* \right)$$

– $\widehat{\Psi}$ retains Majorana feature of describing excitations that carry no charge: current density for Ψ vanishes due to pseudo-reality constraint

N.B. The Majorana/reality properties are obscured by the representation of the Dirac matrices employed in presenting the Hamiltonian h. One may pass to the Majorana representation by a unitary transformation in which Hamiltonian is purely imaginary and $C = I$ so that pseudo reality condition becomes reality condition.

Topological structure

Case of homogeneous order parameter:

energy eigenvalue $E = \pm\sqrt{(k \pm \mu)^2 + m^2}$; no zero energy ($\triangle e^{-i\omega} = m$; constant phase is removed and m is real constant of indefinite sign.)

At $\mu = 0$ energy is doubly degenerate; degeneracy occurs because h commutes with

$$S = \begin{pmatrix} 0 & e^{i\omega}\sigma^3 \\ e^{-i\omega}\sigma^3 & 0 \end{pmatrix} \Rightarrow \; h' \equiv S\,h = \Sigma_a n^a \quad (a = 1, 2, 3)$$

$$n^i = k^i\,(i = 1, 2) \text{ and } n^3 = m$$

A further unitary transformation shows that Σ satisfies SU (2) algebra.

$$U^{-1}\,\Sigma_a\,U = \begin{pmatrix} \sigma^a & 0 \\ 0 & \sigma^a \end{pmatrix}$$

Topological current in momentum space:

$$K^\mu = \frac{1}{8\pi}\, \varepsilon^{\mu\alpha\beta}\, \varepsilon_{abc}\, \widehat{n}^a\, \partial_\alpha\, \widehat{n}^b\, \partial_\beta\, \widehat{n}^c \quad (\widehat{n} \equiv \mathbf{n}/|\mathbf{n}|)\,,$$

$$\mathcal{N} = \int d^2k\, K^0(\mathbf{k}) = \frac{1}{8\pi} \int d^2k\, \frac{m}{(k^2+m^2)^{3/2}} = \frac{m}{2|m|}$$

$\sim$ mapping of $R^{(2)}$ to $S^{(2)}$: $\widehat{n}^a = (k\cos\varphi, k\sin\varphi, m)/\sqrt{k^2+m^2}$

When k begins at $k=0, \widehat{n}^a$ is at the north or south pole, as k ranges to ∞, $\widehat{n}^a$ covers a hemisphere (upper or lower) and ends at the equator of $S^{(2)}$. Thus only one half of $S^{(2)}$ is covered.

– evidence that the model belongs to a topologically non-trivial class.

$\Rightarrow$ topologically protected zero modes exist in the presence of a vortex.

In the presence of a Single Vortex Order Parameter:

$$\triangle(\mathbf{r}) = v(r)e^{i\theta}$$

Energy eigenvalue equation possesses an isolated zero energy mode:

$$\psi_0^v = N \begin{pmatrix} J_0(\mu r)\,\text{exp}\{-i\pi/4 - V(r)\} \\ J_1(\mu r)\,\text{exp}\{i(\theta + \pi/4) - V(r)\} \end{pmatrix}$$

N : real constant
$V'(r) = v(r)$

$$\Psi_0^v = \begin{pmatrix} \psi_0^v \\ i\sigma^2\,\psi_0^{v*} \end{pmatrix} \qquad C\Psi_0^{v*} = \Psi_0^v$$

There are also continuum modes.

$$\hat{\Psi} \equiv \sum_{E>0} \left(a_E\, e^{-i\,E\,t}\,\Phi_E + a_E^\dagger\, e^{i\,E\,t}\,C\Phi_E^*\right) + A\,\sqrt{2}\,\Psi_0^v$$

A is the operator for the zero mode and is Hermitian $A = A^\dagger$, anti-commutes with $(a_E, a_E^\dagger)$ and obeys $\{A, A\} = 2\,A^2 = 1$.

How is A realized on states ?

Two possibilities:

1) two one-dimensional realization : take the ground state to be an eigenstate of A; two eigenvalues $\pm\frac{1}{\sqrt{2}} \Rightarrow$ two ground states $|0_+\rangle$ and $|0_-\rangle$

- Two towers of states built upon them; no local operator connects them.

$$a^\dagger_E\, a^\dagger_{E\prime}\, a^\dagger_{E\prime\prime} \ldots |0\pm\rangle$$

- Fermion parity is lost since A is a fermionic operator with

$$\langle 0_+|\, A\, |0_+\rangle = \frac{1}{\sqrt{2}} \qquad \text{(similarly for } |0_-\rangle\text{)}$$

2) two-dimensional realization: vacuum is doubly degenerate − call one **bosonic** state $|b\rangle$, the other **fermionic** $|f\rangle$ and A connects the two:

$$A\,|b\rangle = \frac{1}{\sqrt{2}}\,|f\rangle \qquad A\,|f\rangle = \frac{1}{\sqrt{2}}\,|b\rangle$$

- two towers of states built on $|b\rangle$ and $|f\rangle$ are connected by A!

observe: $|0_+\rangle = \frac{1}{\sqrt{2}}\,(|b\rangle + |f\rangle) \qquad |0_-\rangle = \frac{1}{\sqrt{2}}\,(|b\rangle - |f\rangle)$

These states violate fermion parity.

Which realization to choose?

To establish fermion parity preserving realization in the presence of a vortex we next consider **vortex/anti-vortex background**.

- There is no zero mid-gap mode – it splits into two low lying states with opposite energy; when vortex and anti-vortex are separated by a large distance $R, \varepsilon \approx \pm e^{-mR}$, where m is the asymptotic value of $v(r)$ as $r \to \infty$.

- Quantum field

$$\widehat{\Psi} = \widehat{\Psi}_{\text{cont}} + a_\varepsilon \, e^{-i\varepsilon t} \, \Psi_\varepsilon^{v\bar{v}} + a_\varepsilon^\dagger \, e^{i\varepsilon t} \, C\Psi_\varepsilon^{v\bar{v}}$$

  Fock space spectrm

  low-lying state:

$$a_\varepsilon \, |\Omega\rangle = 0 \qquad a_\varepsilon^\dagger \, |\Omega\rangle = |f\rangle$$
$$a_\varepsilon \, |f\rangle = |\Omega\rangle \qquad a_\varepsilon^\dagger \, |f\rangle = 0$$

Remaining states:

$$a_E^\dagger \; a_{E'}^\dagger \cdots |\Omega\rangle$$
$$a_E^\dagger \; a_{E'}^\dagger \cdots |f\rangle$$

Limit $R \to \infty$: $\psi^{v\hat{v}}_{\pm\varepsilon} \xrightarrow[\varepsilon \to 0]{} \psi^v_0$

$$\begin{aligned} \hat{\Psi} &= \hat{\Psi}_{\text{cont}} + \frac{(a_\varepsilon + a_\varepsilon^\dagger)}{\sqrt{2}} \sqrt{2}\,\psi^v_0 \\ &= \hat{\Psi}_{\text{cont}} + A\sqrt{2}\,\psi^v_0 \end{aligned}$$

$$\begin{cases} A = A^\dagger \\ |\Omega\rangle \equiv |b\rangle \end{cases} \qquad \{A, A\} = 1$$

$$\Rightarrow \; A\,|b\rangle = \frac{1}{\sqrt{2}}\;|f\rangle \qquad A\,|f\rangle = \frac{1}{\sqrt{2}}\;|b\rangle$$

Note:
Above discussion is somewhat qualitative since no explicit solutions are available in the background of vortex/anti-vortex. However, in 1-d, with Majorana fermions in the presence of kink/anti-kink background, one may solve equations explicitly and verify above statement. [See G. W. Semenoff and P. Sodano (2006) – these authors do not definitely select between the one- and two-dimensional representations of the zero mode algebra.]

Remaining Question : Who will discover Majorana fermions first, condensed matter physicists or particle physicists ?