Lieb-Liniger model

A heart of cold, soft and cool collection of atoms

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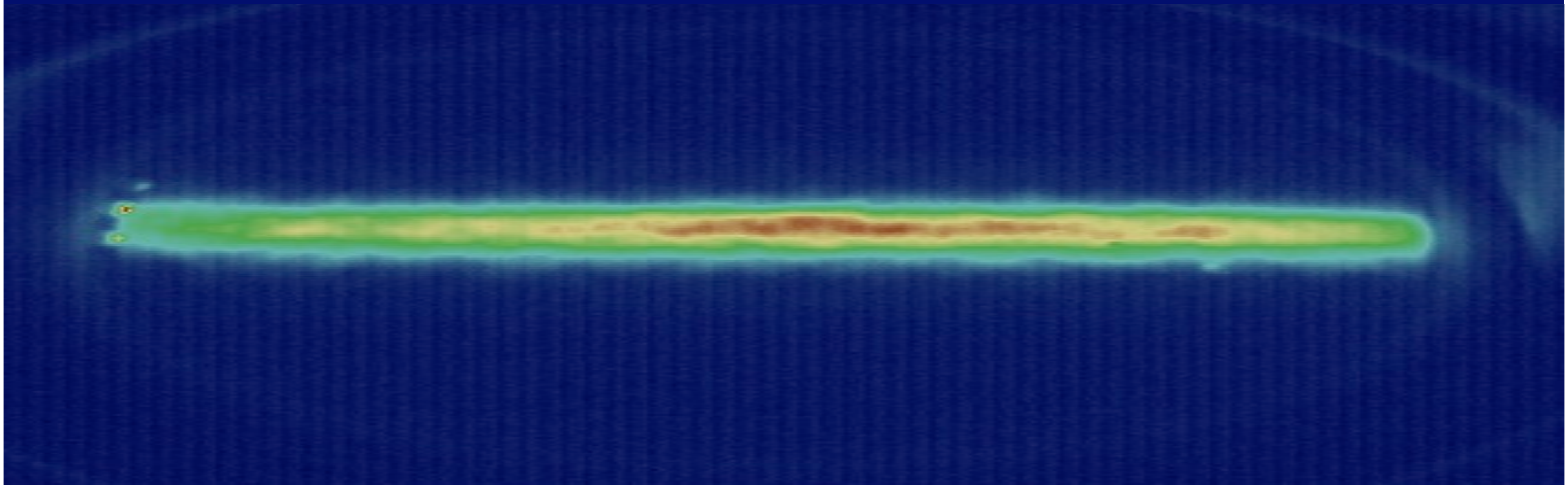

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Expectation Values in the Lieb-Liniger Bose Gas

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We present a novel method to compute expectation values in the Lieb-Liniger model both at zero and finite temperature. These quantities, relevant in the physics of one-dimensional ultracold Bose gases, are expressed by a series that has a remarkable behavior of convergence. Among other results, we show the computation of the three-body expectation value at finite temperature, a quantity that rules the recombination rate of the Bose gas.

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Correlation functions are key quantities in quantum interacting systems; not only do they fully encode the dynamics but they are also directly related to various susceptibilities and response functions. For these reasons, there has always been an intense search to find the most efficient ways to compute them. The task is notoriously difficult, even if the system is integrable. A typical but significant example is provided by the Lieb-Liniger (LL) model [1] that describes the low-temperature properties of one-dimensional interacting Bose gases; although it can be solved through Bethe ansatz equations, the explicit computation of its correlation functions is a long-standing problem [2,3]. The interest in the computation of correlation functions in the LL model is obviously not only theoretical. In a series of recent experimental achievements strongly interacting ultracold bosons have been confined within waveguides by nearly one-dimensional potentials that tightly trap the particle motion in the two transverse directions while leaving it free in the third axial direction [4–6]; the coupling of these bosons with the external world can be made so weak that their behavior is very well described by the LL model. Through interference (or eventually *in situ*) experiments, several quantities can be detected both at zero and finite temperature: the time duration of experiments depends on the three-body recombination rate, which is proportional to local three-body expectation values [7]. Many important general questions of quantum many-body physics can be studied in such a highly controllable setup: dynamical properties concerning the absence (or not) of thermalization [4,8], for instance, or the behavior of integrable quantum systems when small nonintegrable perturbations (e.g., three-body interactions and/or a weak external trapping potential) are switched on [9].

Over the years several theoretical quantities of the LL model have been computed by means of different techniques [10–19]. In this Letter we present a compact and general way to determine the expectation values of its local operators. The method takes advantage of an exact mapping between a relativistic integrable massive model—the sinh–Gordon (ShG)—and the LL model: in a proper nonrelativistic limit of the ShG model, both its S-matrix and Lagrangian coincide with those of the LL model. Since the S matrix of an integrable relativistic model fixes the exact matrix elements of all operators of the theory (and for the ShG model these matrix elements are all known [20,21]), the correspondence between the two models opens the way to computing the corresponding quantities of the LL model in a very direct way. As shown below, this method provide[s] a remarkable simplification of the problem. Its implemen[...] tation actually requires us to take into account an ad[...] tional aspect of the problem: while in the ShG model [...] correlation functions refer to the vacuum (i.e., the s[...] without any particles), in the LL model they relate ins[...] to its ground state at a finite density. This aspect, how[...] can be successfully overcome by the thermodyna[...] Bethe ansatz (TBA) formalism developed in [22], [...] has the additional convenience of being applicable [...] well both at zero and finite temperature. In this wa[y...] able to compute not only the zero temperature ex[...] values but also their finite temperature expressio[...]

The LL-ShG mapping.—The LL Hamiltonian [...] teracting bosons of mass m in one dimension i[...]

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\lambda\sum_{i<j}\delta(x_i - \ldots$$

The corresponding nonrelativistic field the[...] is the quantum nonlinear Schrödinger m[...] employs the complex field ψ and the La[...]

$$\mathcal{L} = -\frac{\hbar^2}{2m}|\nabla\psi|^2 + i\frac{\hbar}{2}\left(\psi^\dagger\frac{\partial\psi}{\partial t} - \frac{\partial\psi^\dagger}{\partial t}\ldots\right.$$

The effective coupling constant of th[...] by the dimensionless parameter $\gamma = \ldots$ [...] 0 is the coupling entering the Hami[...] N/L is the density of the gas (L is th[...] Temperatures are usually express[...] perature $T_D = \hbar^2 n^2/2mk_B$ of th[...] $\tau = T/T_D$. The two-body elastic [...] is [1,10]

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One-dimensional Lieb-Liniger Bose gas as nonrelativistic limit of the sinh-Gordon model

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The repulsive Lieb-Liniger model can be obtained as the nonrelativistic limit of the sinh-Gordon model: all physical quantities of the latter model (S-matrix, Lagrangian, and operators) can be put in correspondence with those of the former. We use this mapping, together with the thermodynamical Bethe ansatz equations and the exact form factors of the sinh-Gordon model, to set up a compact and general formalism for computing the expectation values of the Lieb-Liniger model both at zero and finite temperatures. The computation of one-point correlators is thoroughly detailed and when possible compared with known results in the literature.

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I. INTRODUCTION

The physics of one-dimensional interacting bosons is well captured by the Lieb-Liniger (LL) model [1]. Despite its deceptive simplicity, this model has become a paradigmatic example of quantum integrable systems since it proved to have a remarkable richness [2]: its Bethe ansatz equations, for instance, can be explicitly derived and used to study its equilibrium properties at zero and finite temperatures [3]. The explicit analysis of the weak-to-strong coupling crossover of this model has also set a precise benchmark for approximate many-body techniques [1,2]. The efforts done for the computation of the correlation functions of the LL model also greatly stimulated the development of new and general formalisms, such as the quantum inverse scattering method [4] or the bosonization approach [5].

Nowadays a renewed interest in the LL model has been triggered by its accurate experimental realization [6–9]. In quasi-one-dimensional traps, the excitations in the transverse directions are effectively frozen and, moreover, the coupling of the ultracold bosons to the external environment can be made very weak [10]. These recent experimental advances have opened new perspectives in the field of strongly correlated quantum systems. In such a highly controllable set-up it is, in fact, possible to thoroughly investigate problems of a general nature concerning quantum extended systems, such as the dynamics of integrable systems in the presence of small nonintegrable perturbations [11] (e.g., three-body interactions and/or a weak external trapping potential), the issue of thermalization in quantum integrable and nonintegrable systems [12], and the behavior of various susceptibilities and response functions.

The key quantities to answer all these questions are the correlation functions of the LL model. Despite the integrability of the model, their explicit computation turned out to be an interesting theoretical challenge. For this reason many different approaches were developed over the years to tackle different aspects of this difficult problem: a partial list includes bosonization (which gives the correct long-distance behavior of correlation functions) [5,13,14], quantum Monte Carlo simulations [15], algebraic Bethe ansatz [16], analytical-numerical methods based on the exact Bethe ansatz solution [17,18], Bogoliubov weak- [19] and strong-coupling methods [20], renormalization group [21], numerical results using stochastic wave functions [22], and imaginary time simulations [23,24]. Exact results based on Yang-Yang equations and the Hellmann-Feynman theorem are presently available for local two-body correlations [20,25], while the local three-body correlations were determined at zero temperature in Ref. [26].

An alternative method was recently proposed to compute expectation values in the LL model [27]: it exploits a different route from all the previous approaches for it is based on an exact mapping between the nonrelativistic LL model and the relativistic integrable sinh-Gordon (sh-G) model. This proposal not only provides a remarkable simplification of the problem, but applies equally well both at zero and finite temperatures. In summary, the logical steps on which the method is based are the following:

1. Due to the relativistic invariance and quantum integrability of the sh-G model, it is possible to set up functional equations [28,29] for the matrix elements of its local operators on the asymptotic states—known as *form factors*—and to find their exact solutions [30,31].

2. The finite temperature and finite density effects of the sh-G model can also be controlled by solving the thermodynamical Bethe ansatz (TBA) equations [32,33].

3. In a proper nonrelativistic limit of the sh-G model, all quantities of this theory—S matrix, Lagrangian, form factors, TBA equations, and so on—reduce to those of the LL model and therefore can be used to establish an explicit mapping between the two models. In particular, from the exact and known expressions of the form factors of the sh-G model we can explicitly obtain the matrix elements of the operators of the LL model we are interested in.

4. To actually compute the LL correlation functions we have to take into account another aspect of the problem, that is, that the LL correlation functions refer to the ground state of the gas at a finite density and at a finite temperature, while those of the sh-G model refer to the vacuum state (i.e., the state without any particles). This apparent difficulty can be, however, readily overcome by using the LeClair-Mussardo formalism [34] that, as a matter of fact, is based on the same quantities mentioned previously (i.e., the form factors and the TBA equations).

In this article we provide the details of the mapping between the sh-G and the LL models established in Ref. [27], presenting

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Overlook and plan of the seminar

- Bird-eye view on Lieb-Liniger model
- Wave function and S-matrix
- Correlation functions in the LL model
- Sh-Gordon model and mapping to the LL model
- Form Factor theory
- Thermal and finite density average (LM formalism)

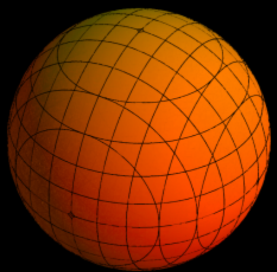
Experimental setup (I)

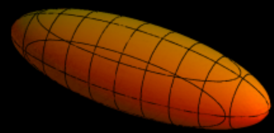
Magnetic harmonic potential: $V(x,y,z)=\frac{1}{2}m\left(\omega_x^2 x^2+\omega_y^2 y^2+\omega_z^2 z^2\right)$

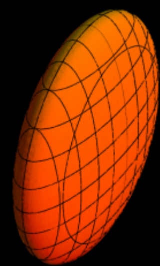
$\omega_x=\omega_y=\omega_z$

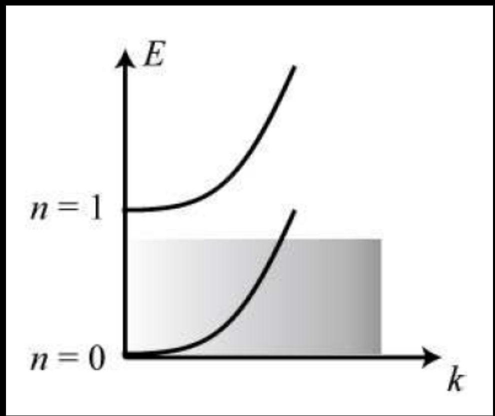
$\omega_x \ll \omega_y=\omega_z$

$\omega_x \gg \omega_y=\omega_z$









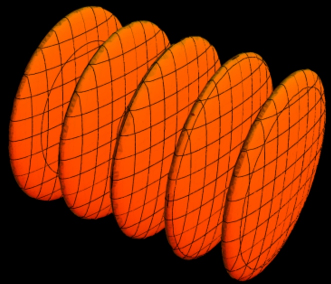
1D condition: <u>frozen</u> transverse degrees of freedom

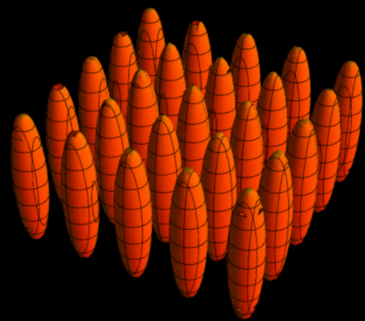
Experimental setup (II)

Using optical lattices:

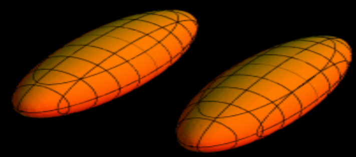
$$V = V_0 \cos^2(kx)$$

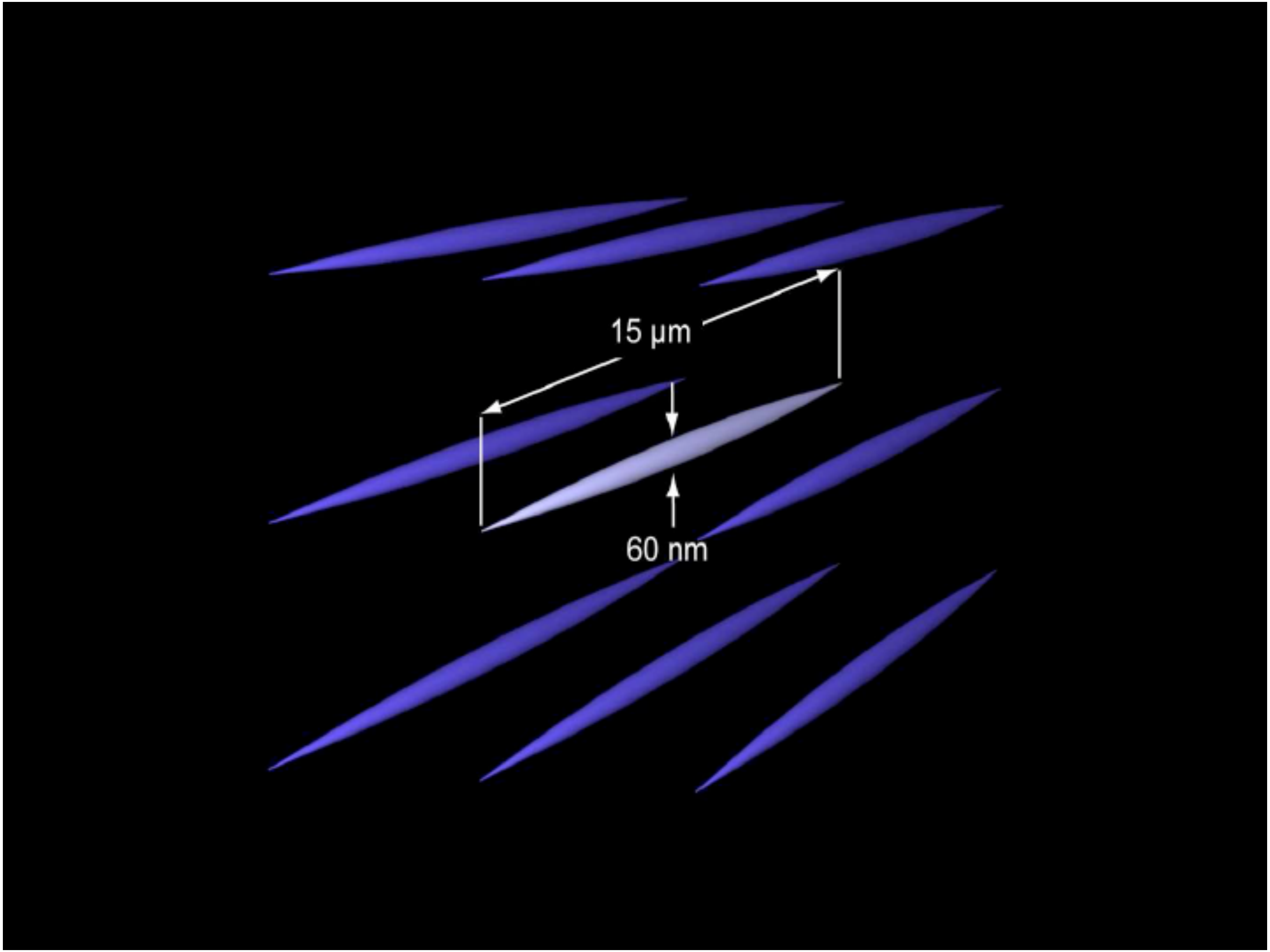
$$V = V_0\left[\cos^2(kx) + \cos^2(ky)\right]$$





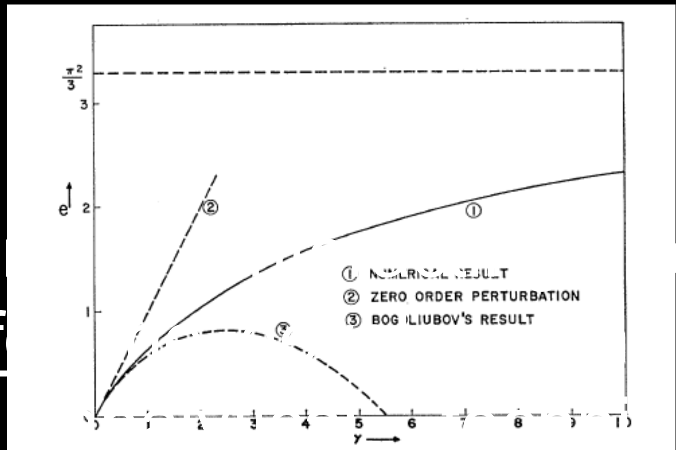
Other configurations (like ladders or coupled cigars) are as well possible:




15 µm
60 nm

Low-dimensional Bose gases

- Failure of mean-field approaches



- 1D Hamiltonia osons (Lieb-Liniger model) called f of non-perturbative, numerical and ues and used as benchmark for approximate methods

- In the last 10 years: Experimentally realized (with tunable parameters) with ultracold atoms

Lieb-Liniger Hamiltonian

N interacting bosons in 1D:

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\lambda\sum_{i<j}\delta\left(x_i - x_j\right)$$

One non-trivial coupling constant:

$$\gamma = \frac{2m\lambda}{\hbar^2 n}$$

density

Temperature in units of the degeneracy temperature:

$$k_B T_D = \frac{\hbar^2 n^2}{2m}$$

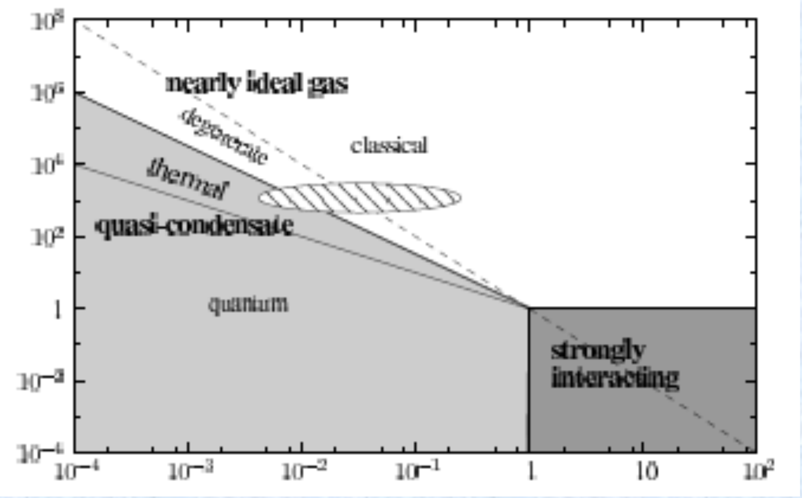
Coupling controllable from the 3D setup:

$$\lambda = \frac{\hbar^2 a_{3D}}{m a_\perp^2}\frac{1}{1 - C\, a_{3D}/a_\perp}$$

Lieb-Liniger Hamiltonian (II)

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\lambda\sum_{i<j}\delta\left(x_i - x_j\right)$$

τ



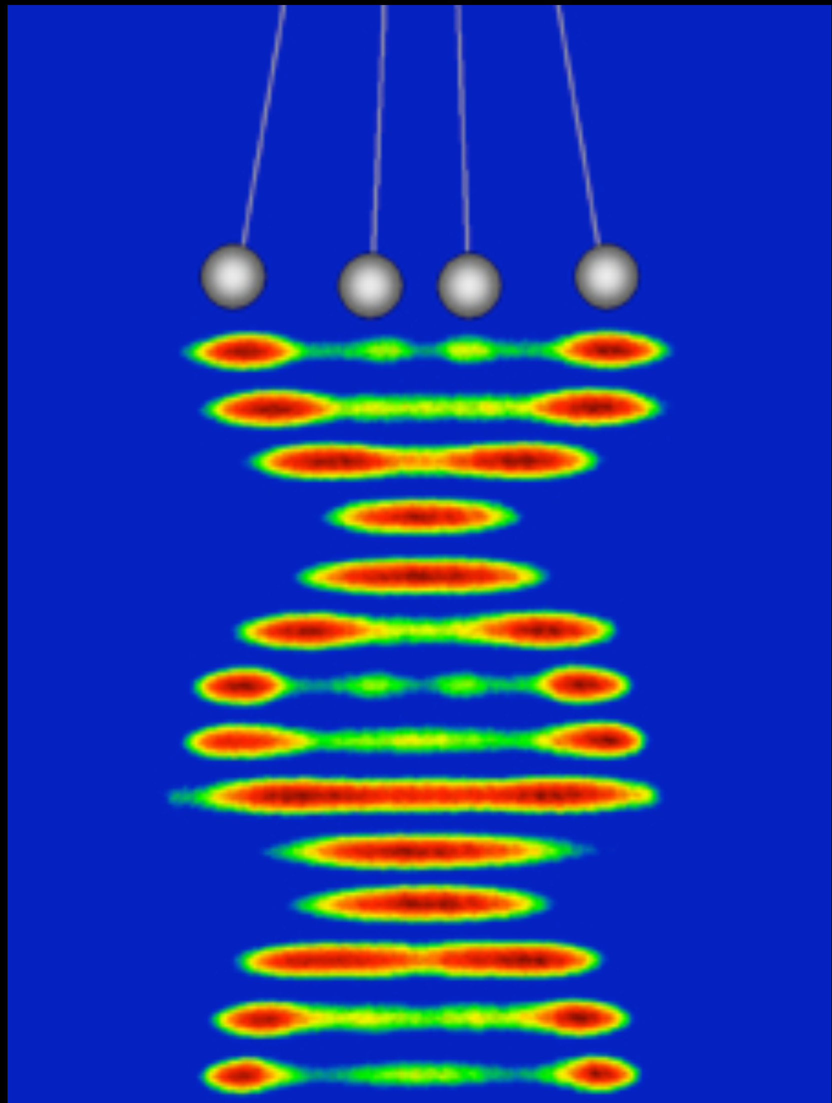
γ

$$\gamma = \frac{2m\lambda}{\hbar^2 n}$$

$$\tau = \frac{T}{T_D}$$

Newton cradle of cold atoms

Kinoshita et al., Nature (2006)



Correlation functions

Despite the exact solution of the thermodynamics of the model (Lieb-Liniger, Yang-Yang), the computation of its correlation functions turns out to be a **hard problem**

Bosonisation, quantum MC, BA solution, weak and strong coupling expansions, . . .

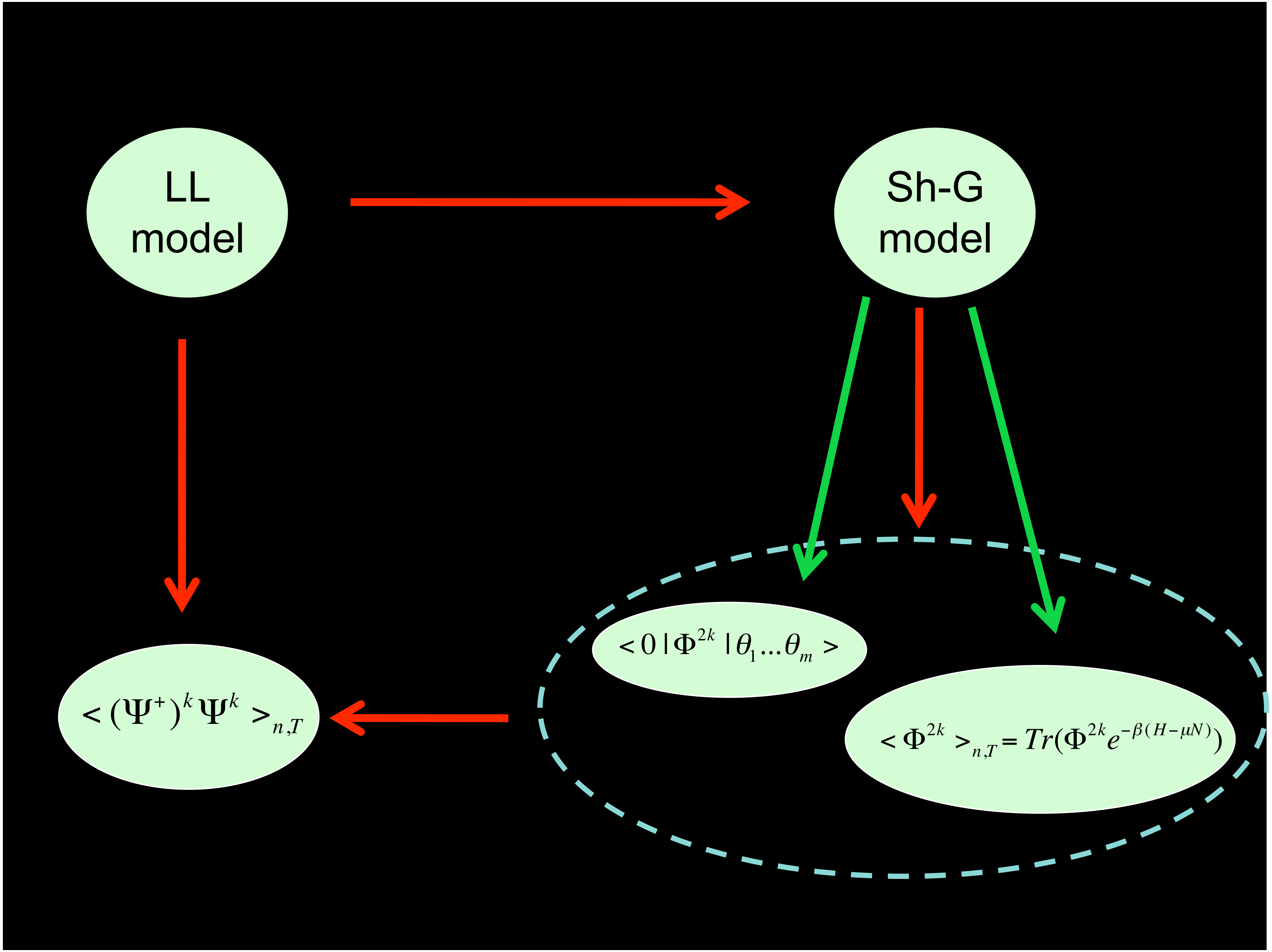
Unified approach

(Kormos, GM, Trombettoni)

- Start from a relativistic integrable QFT and consider the LL model as a suitable non-relativistic limit thereof

- An integrable relativistic integrable QFT has **all** the equations needed to pin down the exact matrix elements

- The correlation functions are then expressed in terms of spectral representation at finite density and temperature



LL model
Sh-G model
$< (\Psi^{+})^{k}\Psi^{k} >_{n,T}$
$< 0 \,|\, \Phi^{2k} \,|\, \theta_1 \ldots \theta_m >$
$< \Phi^{2k} >_{n,T} = Tr(\Phi^{2k} e^{-\beta(H-\mu N)})$

LL model ⟷ Sh-Gordon model

1D Lieb-Liniger and Non-linear Schrodinger

$$H_{NLS} = \int dx \left[\frac{\hbar^2}{2m} \frac{\partial \Psi^+}{\partial x} \frac{\partial \Psi}{\partial x} + \lambda \,|\, \Psi^+ \Psi \,|^2 \right]$$

- The density $\rho(x,t) = \Psi^+(x,t)\Psi(x,t)$ is conserved

- Hence, we can work at fixed number of particles

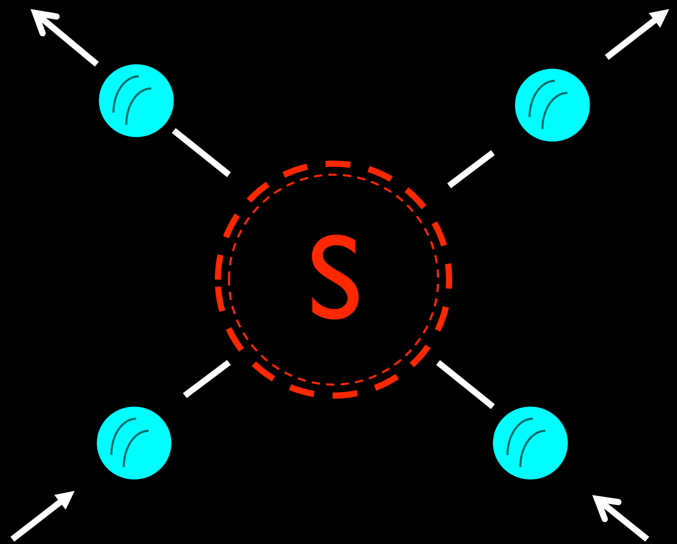
$$| \mathrm{Y}(x_1, \ldots, x_n) >$$
$$= \int da_1 \ldots da_n \mathrm{X}(x_1 \ldots x_n \,|\, a_1 \ldots a_n) \Psi^+(a_1) \ldots \Psi^+(a_n) \,|\, 0 >$$

Bethe Wave Function

$$X(x_1,\ldots,x_n) = \sum_P a(P) e^{i\sum_{j=1}^{N} P(k_j) x_j}$$

$$a(Q) = \frac{k - l - i\frac{2m}{\hbar^2}\lambda}{k - l + i\frac{2m}{\hbar^2}\lambda} a(P)$$

Integrability of LL model

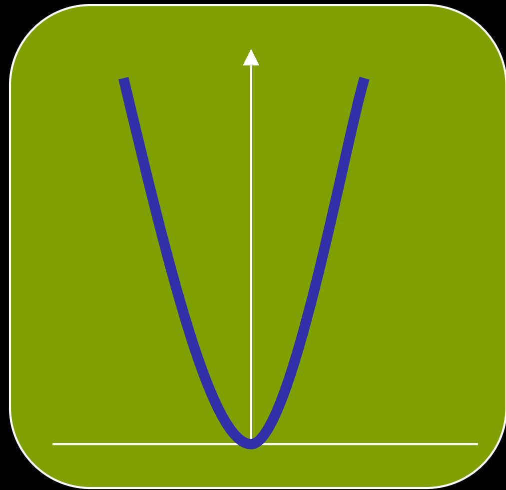


$$S_{LL}(p_1,p_2,\lambda)=\frac{p_1-p_2-i\dfrac{2m}{\hbar^2}\lambda}{p_1-p_2+i\dfrac{2m}{\hbar^2}\lambda}$$

Elastic process

Sinh-Gordon Model

$$L = \frac{1}{2}\left(\frac{\partial\Phi}{c\,\partial t}\right)^2 - \frac{1}{2}(\nabla\Phi)^2 - \frac{\mu^2}{g^2}\left[\cosh(g\Phi) - 1\right]$$



This is the only QFT Z_2 symmetric that is

integrable

Production processes are forbidden

Only elastic scattering

Rapidity variable

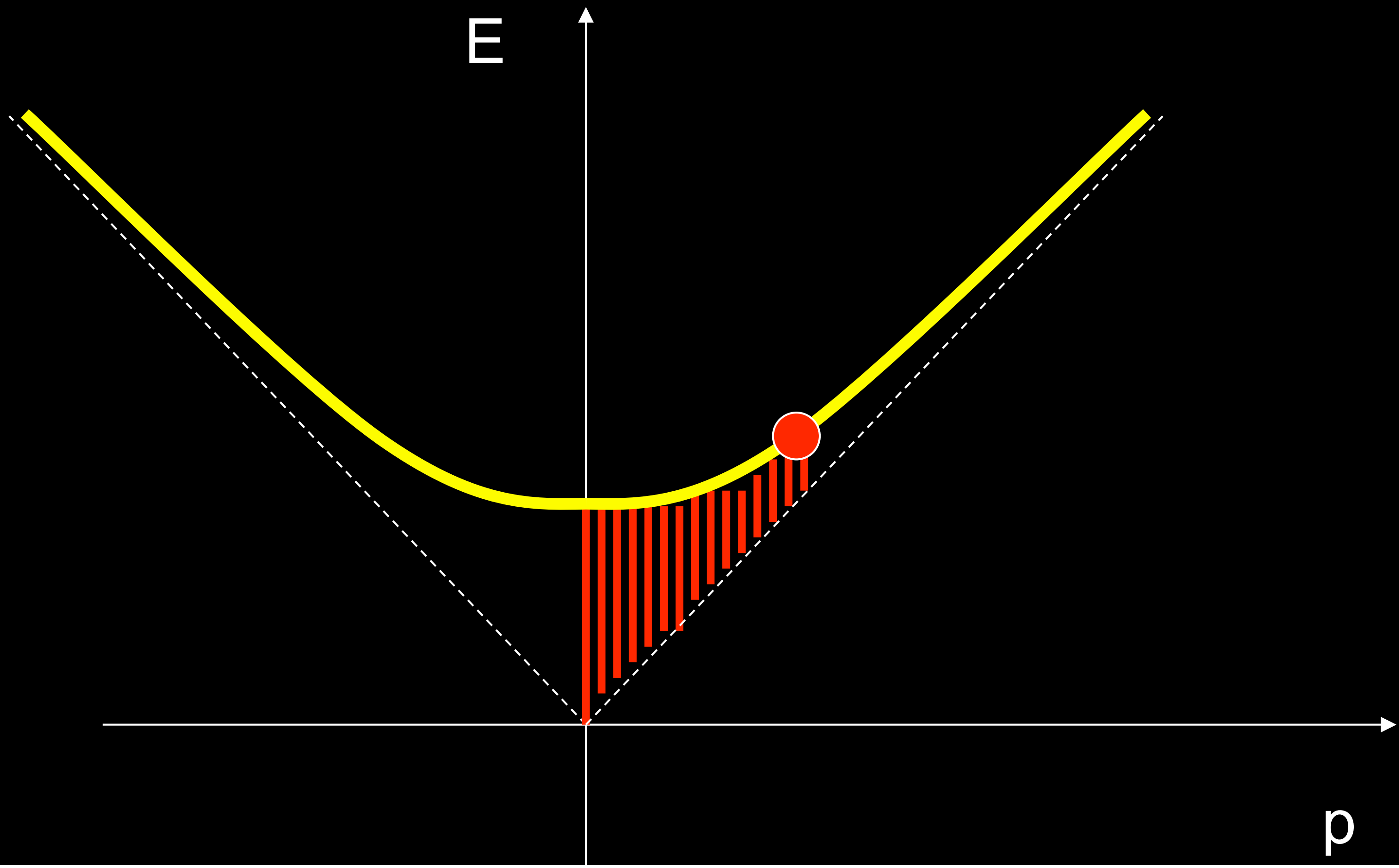
$$E = Mc^2 \cosh\theta \approx Mc^2(1+\theta^2/2)$$

$$E = Mc^2 \cosh(\theta + 2\pi i)$$

$$E^2 - c^2P^2 = M^2c^4$$

$$P = Mc\sinh\theta \approx Mc\,\theta$$

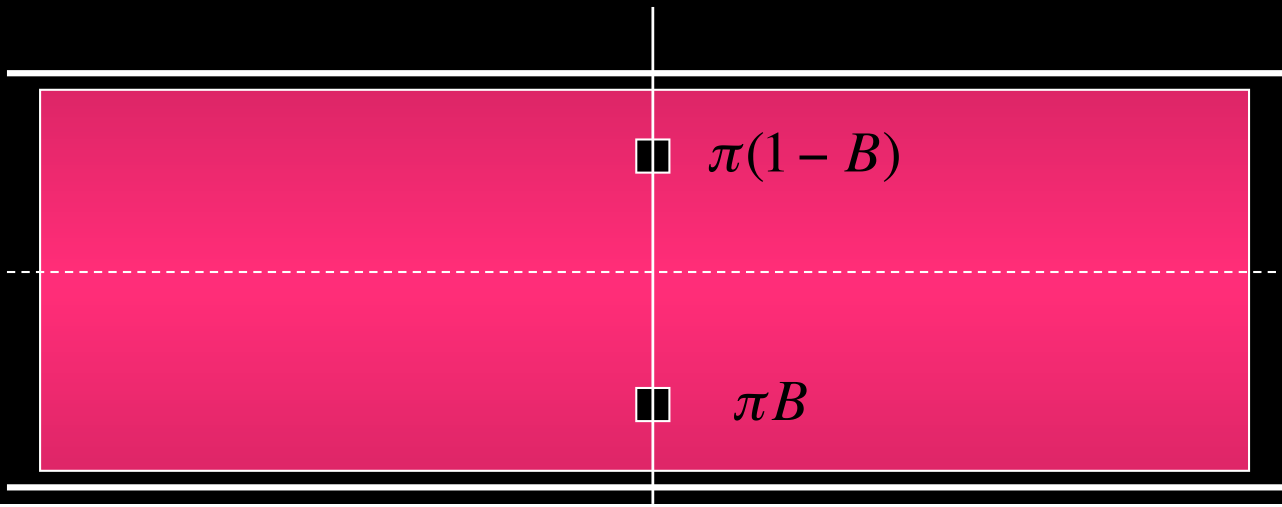
$$P = Mc\sinh(\theta + 2\pi i)$$



Exact Factorizable 2-body S-matrix

$$S_{Sh}(\theta_1,\theta_2,g) = \frac{\tanh\frac{1}{2}\left(\theta_1-\theta_2-i\pi B(g)\right)}{\tanh\frac{1}{2}\left(\theta_1-\theta_2+i\pi B(g)\right)}$$

$$B(g) = \frac{g_*^2}{1+g_*^2} \qquad g_*^2 \equiv \frac{g^2}{8\pi}$$



Infinite-product representation

$$(a) \equiv f_a(\beta) = \frac{\sinh\frac{1}{2}(\beta + i\pi a)}{\sinh\frac{1}{2}(\beta - i\pi a)} \frac{\cosh\frac{1}{2}(\beta - i\pi a)}{\cosh\frac{1}{2}(\beta + i\pi a)} \equiv [a][1-a]$$

Euler's formula

$$\sinh z = z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{k^2\pi^2}\right) = z\prod_{k=1}^{\infty}\left(1 + \frac{z}{ik\pi}\right)\left(1 - \frac{z}{ik\pi}\right)$$

$$[a] = \frac{\sinh\frac{1}{2}(\beta - i\pi a)}{\sinh\frac{1}{2}(\beta + i\pi a)} = \prod_{k=-\infty}^{+\infty}\frac{\beta - i\pi a + 2k\pi i}{\beta + i\pi a + 2k\pi i}$$

Mapping between the Sh-G and LL models

In the double limits

$$c \rightarrow \infty \quad , \quad g \rightarrow 0$$

$$gc \rightarrow \frac{4\sqrt{\lambda}}{\hbar} = fixed$$

the S-matrices of the two models coincide!

$$S_{Sh}(\theta, g) = \frac{\tanh \frac{1}{2}\left(\theta_1 - \theta_2 - i\pi B(g)\right)}{\tanh \frac{1}{2}\left(\theta_1 - \theta_2 + i\pi B(g)\right)} \rightarrow \frac{p_1 - p_2 - i\frac{2m}{\hbar}\lambda}{p_1 - p_2 + i\frac{2m}{\hbar}\lambda} = S_{LL}(p, \lambda)$$

The coincidence of the S-matrices induces an

EXACT mapping

between the operator content of the two theories

Next steps

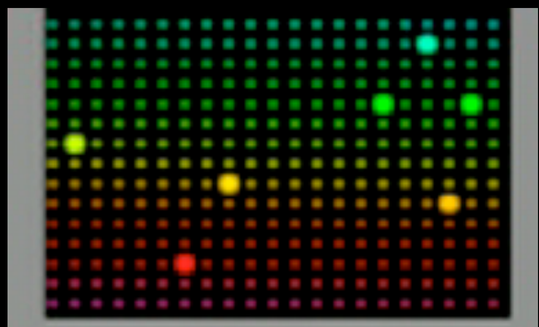
- Establish the exact mapping between the operators

$$\Phi \longleftrightarrow \Psi$$

- Exact expressions of matrix elements (Form Factors) in Sh-G model and their expression in the double limit

$$F^{\Lambda}(\beta_1,\beta_2,\ldots,\beta_n) \equiv <0 \mid \Lambda(0) \mid \beta_1,\beta_2,\ldots,\beta_n>$$

- LeClair-Mussardo formalism to express correlation functions at finite density and finite temperature



Field and Lagrangian

$$\Phi(x,t) = \sqrt{\frac{\hbar^2}{2m}}\left(\Psi(x,t)e^{-i\frac{mc^2}{\hbar}t} + \Psi^+(x,t)e^{+i\frac{mc^2}{\hbar}t}\right)$$

Rule of thumb: neglect all highly time oscillating terms when integrate densities

$$L_{ShG} \rightarrow L_{NLS} = -\frac{\hbar^2}{2m}\frac{\partial\Psi^+}{\partial x}\frac{\partial\Psi}{\partial x} + i\frac{\hbar}{2}\left(\Psi^+\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^+}{\partial t}\right) - \lambda\,|\,\Psi\,|^4$$

What do we want to compute?

Local correlators in LL model

$$<(\Psi^+)^k \Psi^k>_{n,T} = n^k g_k(\gamma,\tau)$$

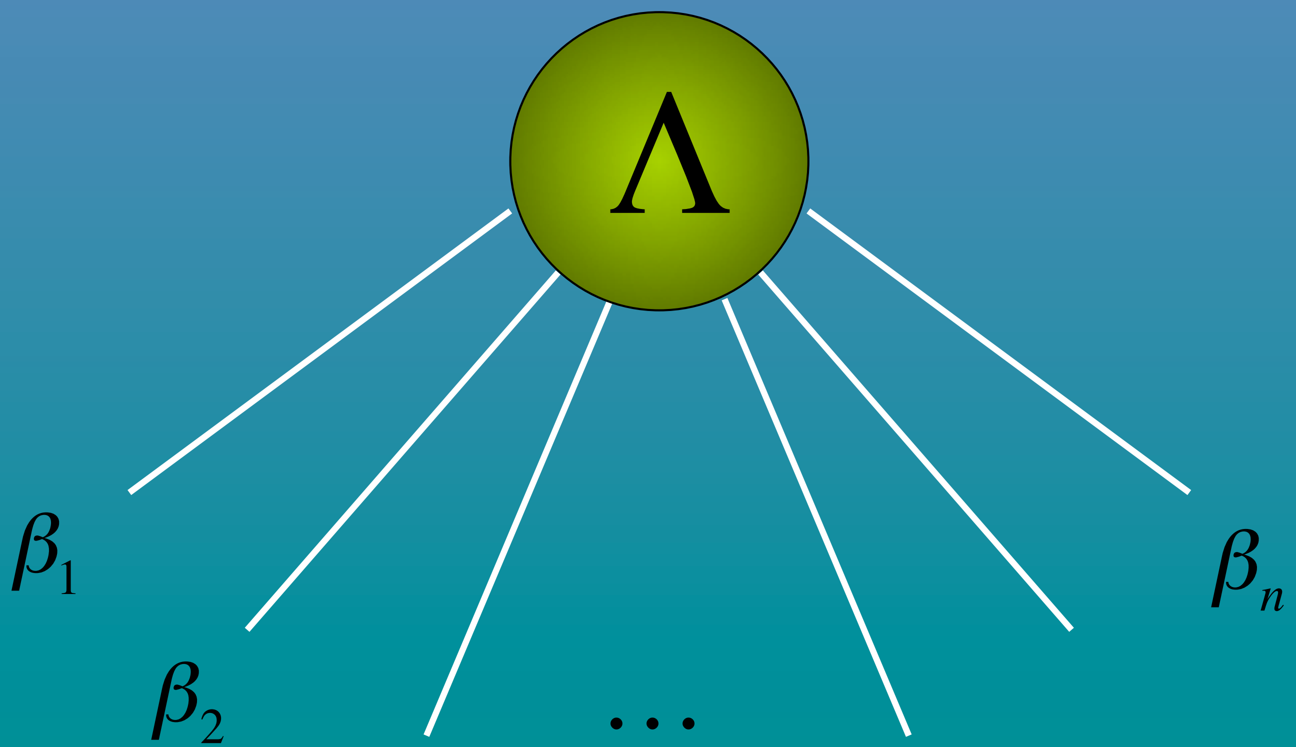
$$\gamma = \frac{2m}{\hbar^2}\frac{\lambda}{n} \qquad \tau = \frac{T}{T_D}$$

$$T_D = \frac{\hbar^2 n^2}{2mk_B}$$

g_3 rules the recombination rate of the gas (i.e. its stability)

Form Factors

$$F^{\Lambda}(\beta_1,\beta_2,\ldots,\beta_n) \equiv < 0 \,|\, \Lambda(0) \,|\, \beta_1,\beta_2,\ldots,\beta_n >$$



<u>Watson’s equations</u>

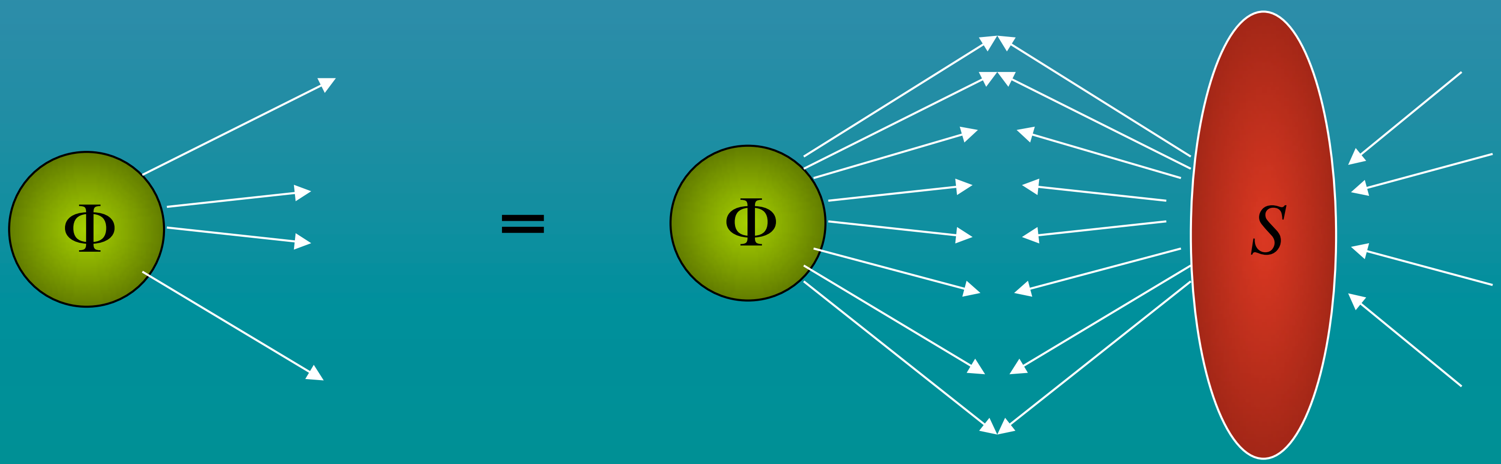
$$\sum_{m=0}^{\infty} |\beta_1 \cdots \beta_m >_{out\ out}< \beta_1 \cdots \beta_m | = 1$$

$$< 0 | \Phi(0) | \beta_1 \cdots \beta_n >_{in} = \sum_{m=0}^{\infty} < 0 | \Phi(0) | \beta_1 \cdots \beta_m >_{out\ out}< \beta_1 \cdots \beta_m | \beta_1 \cdots \beta_n >_{in}$$

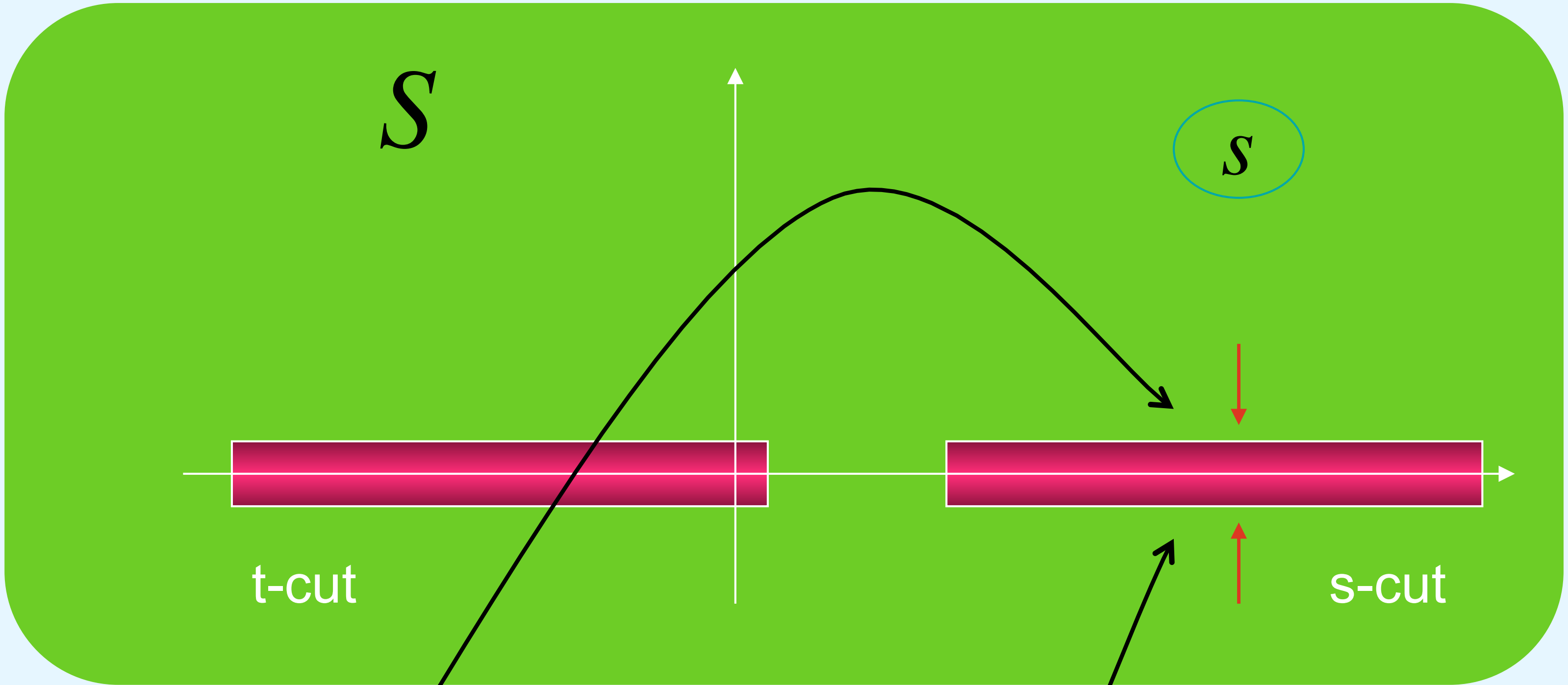
$$= \sum_{m=0}^{\infty} < 0 | \Phi(0) | \beta_1 \cdots \beta_m >_{out} \boxed{S_{n \to n}}$$

The computation reduces to solve a Riemann-Hilbert problem !

But, in integrable theory scattering is elastic!



2-particle Form Factor



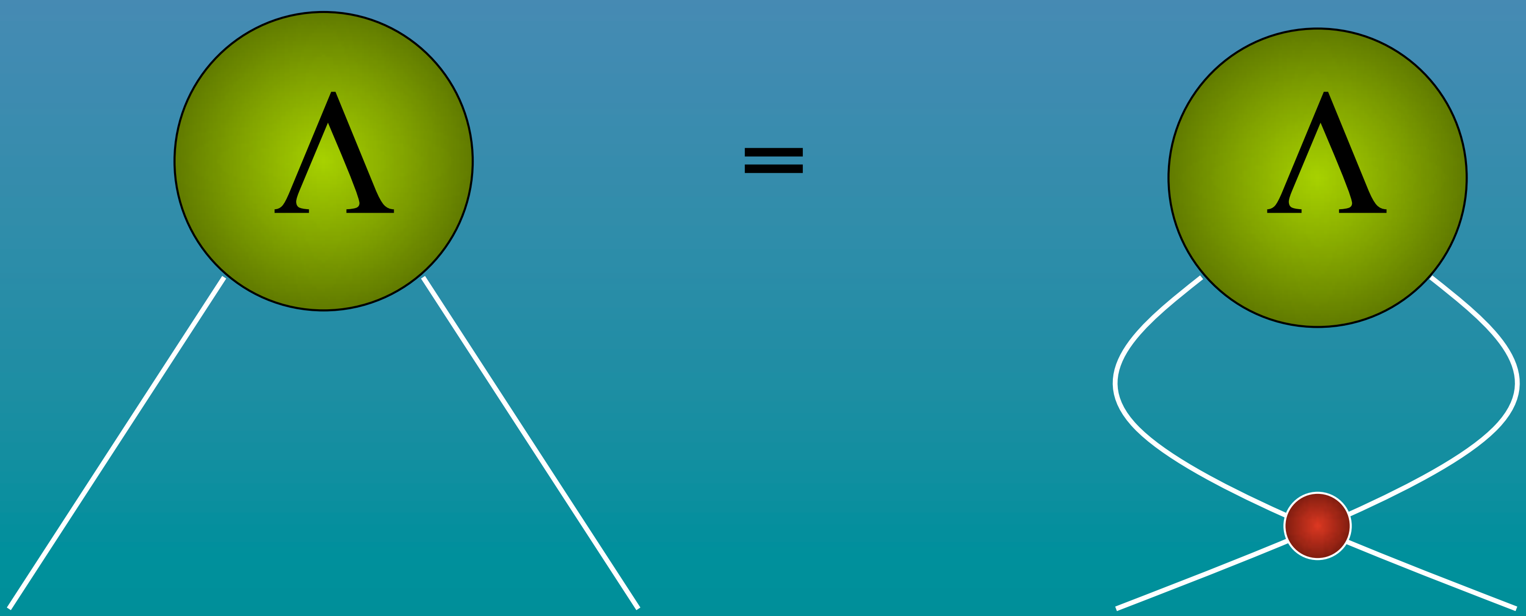
$$F_{ab}(s+i\varepsilon) = \langle 0 | \Phi(0) | A_a(\beta_1) A_b(\beta_2) \rangle_{in}$$

$$F_{ab}(s-i\varepsilon) = \langle 0 | \Phi(0) | A_a(\beta_1) A_b(\beta_2) \rangle_{out}$$

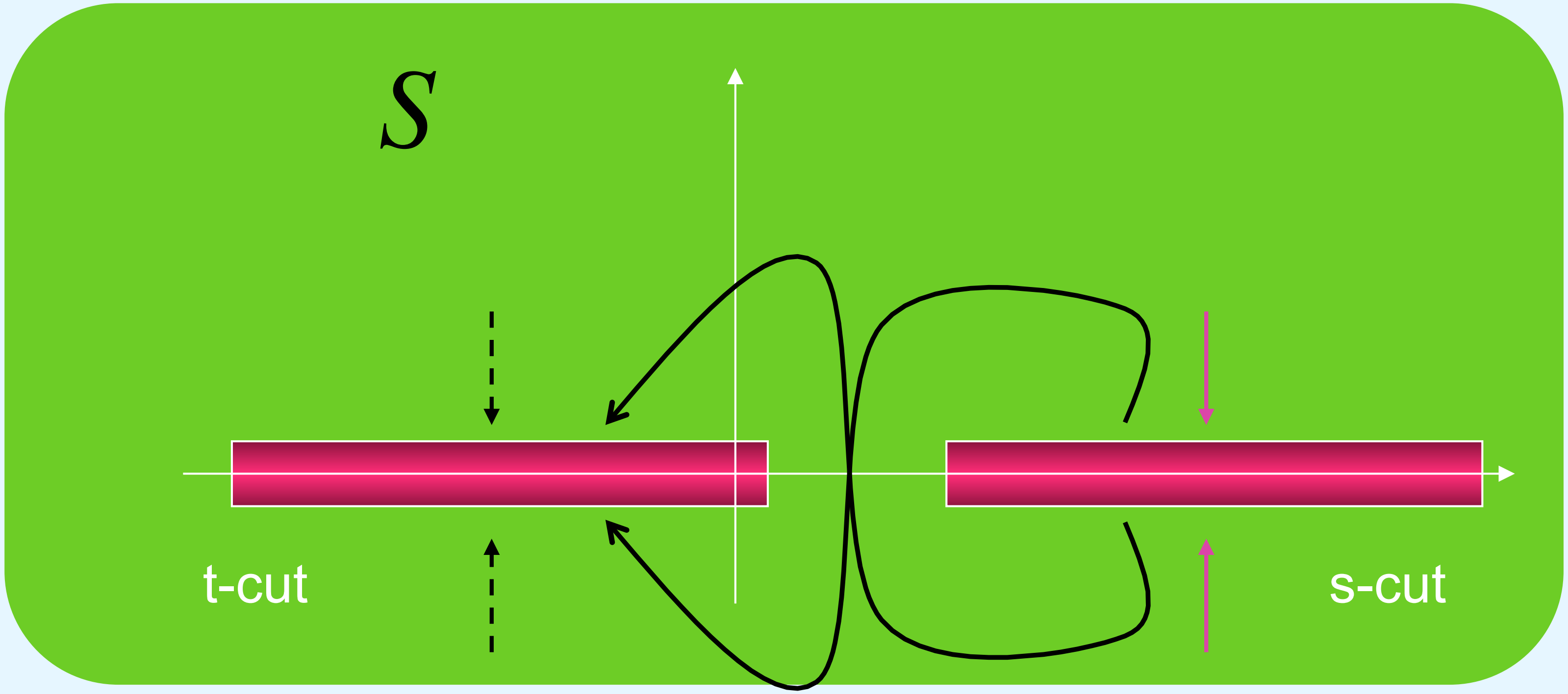
$$F(s+i\varepsilon) = S(s+i\varepsilon) F(s-i\varepsilon)$$

In the rapidity variable

$$F_{ab}(\beta_1-\beta_2)=S_{ab}^{cd}(\beta_1-\beta_2)\,F_{cd}(\beta_2-\beta_1)$$



Behavior at the t-cut



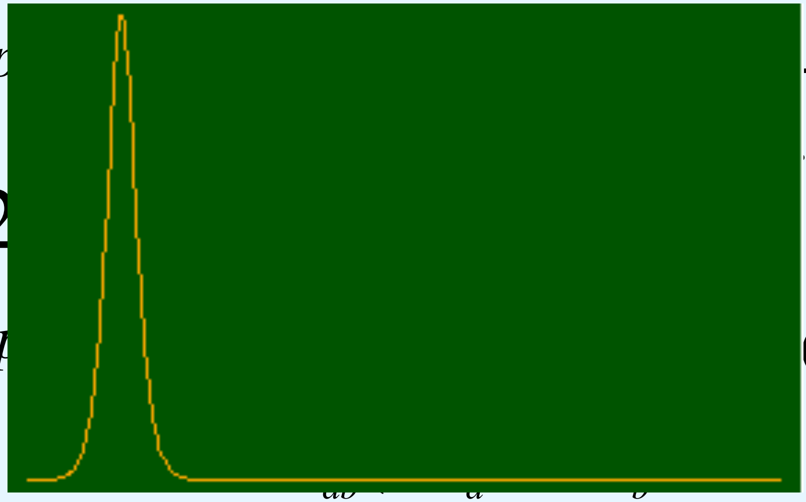
$_{in}< A_{\bar{b}}(p_2)\,|\,\Phi\,|\,A_a(p$ … $-p_2)>_{out}$

$\varepsilon)$

$F(2m_a^2 + 2$ … $2m_b^2 - s - i\varepsilon)$

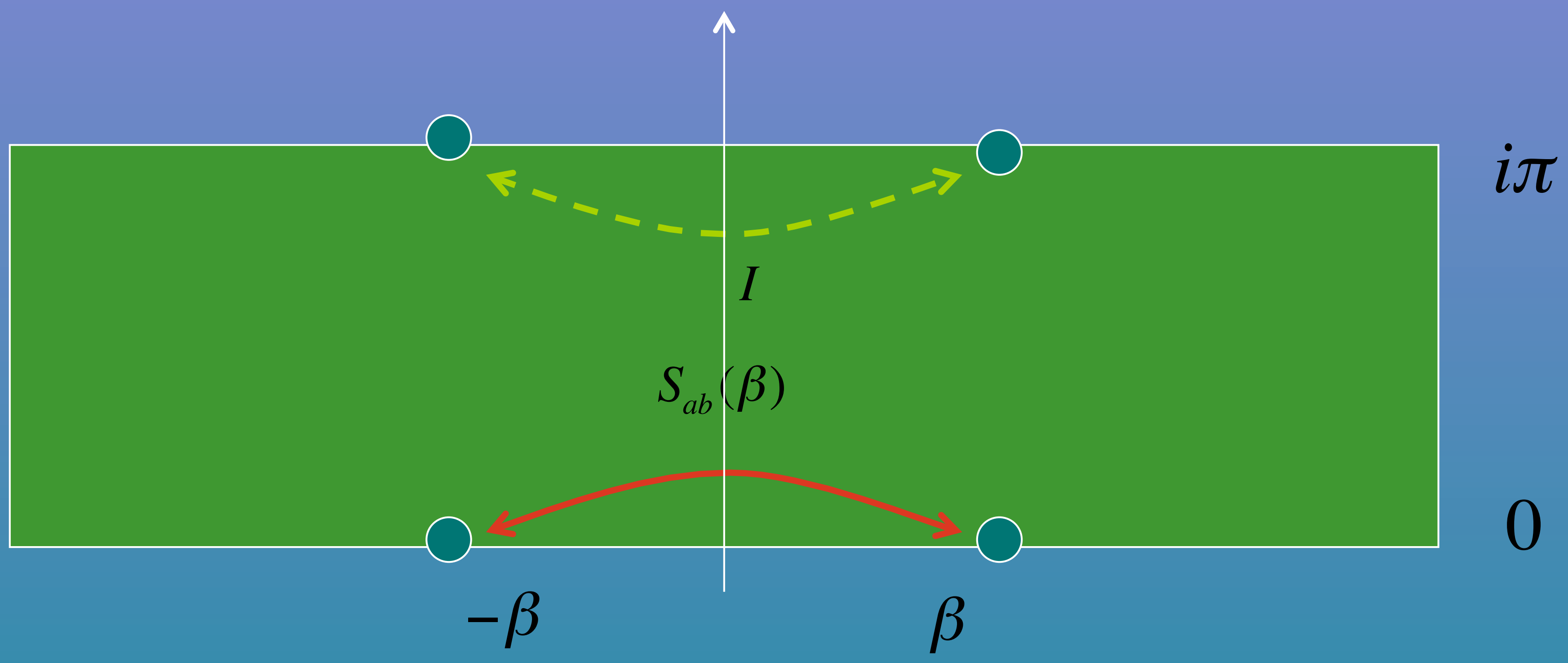
$_{out}< A_{\bar{b}}(p_2)\,|\,\Phi\,|\,A_a(p$ … $(-p_2)>_{in}$

$i\varepsilon)$



In the rapidity variable

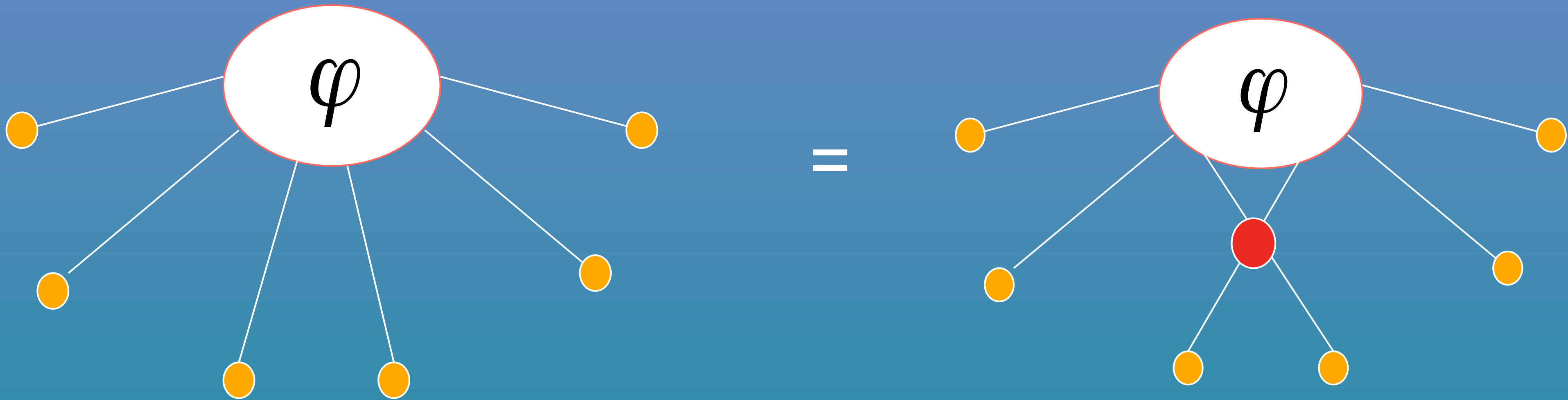
$$F_{ab}(\beta_1 + 2\pi i, \beta_2) = F_{ab}(\beta_2, \beta_1)$$



$$F_{ab}(\beta) = S_{ab}^{cd}(\beta)\, F_{cd}(-\beta)$$

$$F_{ab}(i\pi + \beta) = F_{ab}(i\pi - \beta)$$

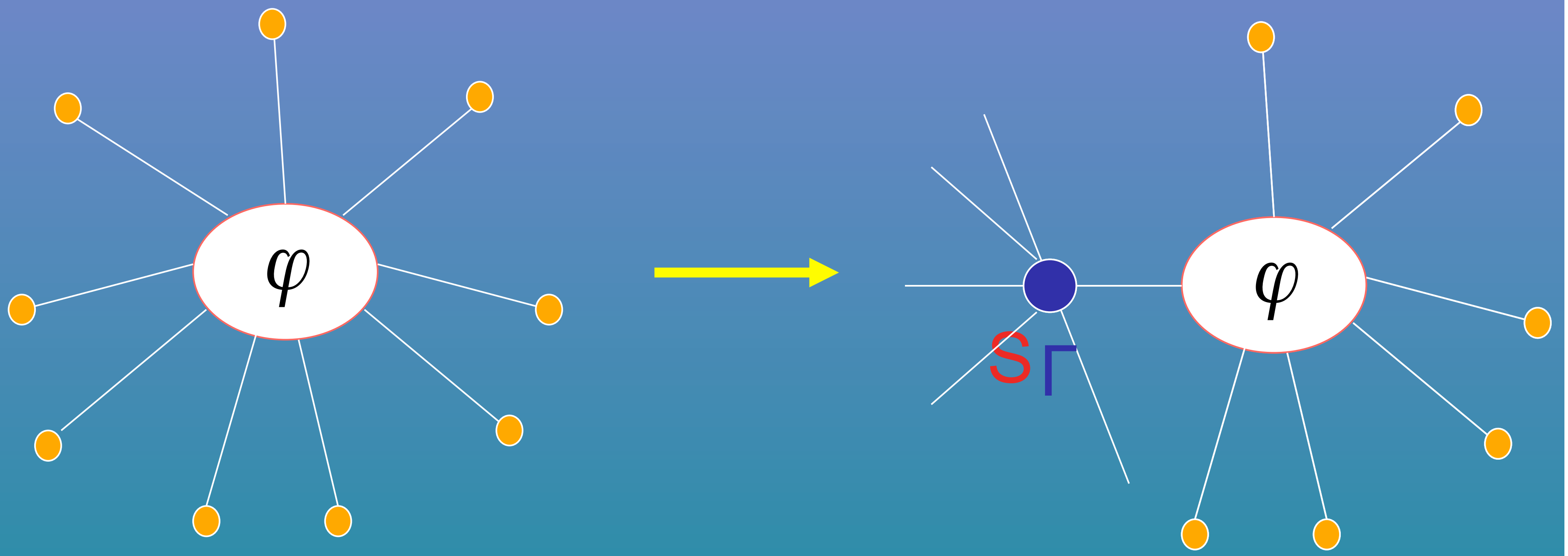
Monodromy properties



$$F(\beta_1,\ldots\beta_i,\beta_{i+1},\ldots\beta_n)=\boxed{S(\beta_i-\beta_{i+1})}\,F(\ldots\beta_{i+1},\beta_i\ldots)$$

$$F(\beta_1+2\pi i,\ldots\beta_i,\beta_{i+1},\ldots\beta_n)=F(\beta_2\ldots\beta_{i+1},\beta_i\ldots\beta_1)$$

Recursive equations

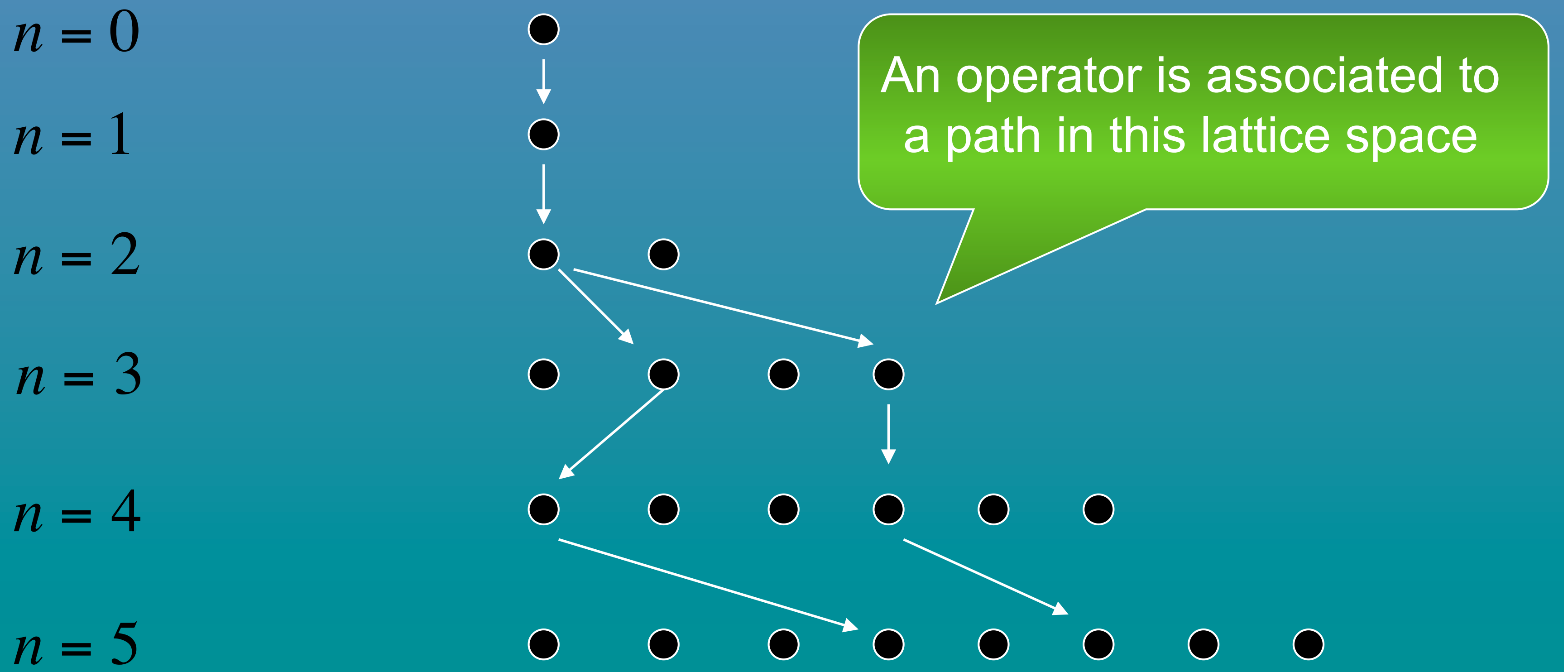


$$F_n \longrightarrow F_{n+2}$$

$$F_n \longrightarrow F_{n-1}$$

Space of operators

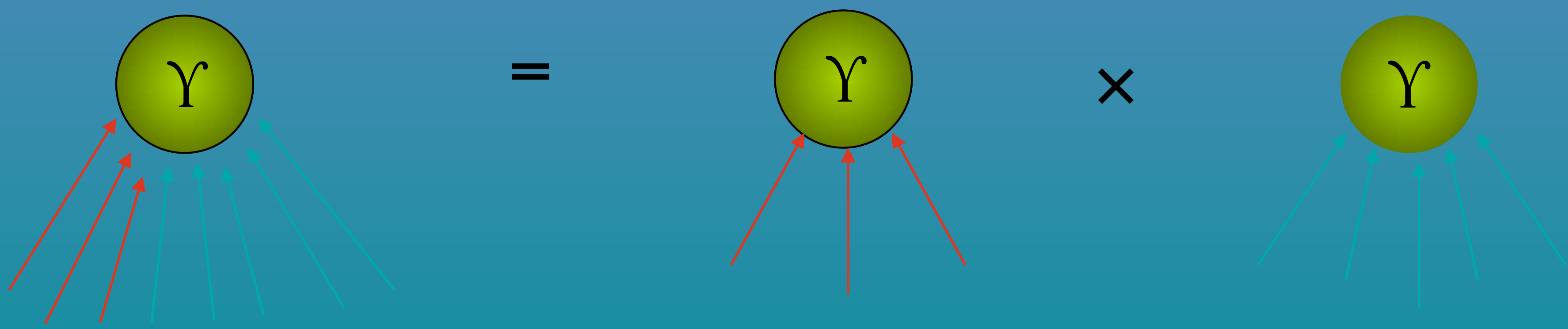
The Form Factors of scalar operators satisfy a set of functional and recursive equations which admit a **finite** number of solutions, given the number n of asymptotic states



Cluster property

$$\lim_{|\Lambda|->\infty} F^{\Phi}_{a_1..a_k..a_n}(\beta_1+\Lambda/2,..,\beta_k+\Lambda/2,\beta_{k+1}-\Lambda/2,..,\beta_n-\Lambda/2) =$$

$$\frac{1}{<\Phi>} F^{\Omega}_{a_1..a_k}(\beta_1...\beta_k)\; F^{\Psi}_{a_{k+1}...a_n}(\beta_{k+1}...\beta_n)$$



Self-clustering operators

$$\Phi^a -> \Phi^b \times \Phi^c$$

$$\Upsilon \rightarrow \Upsilon \times \Upsilon$$

Exact Solutions of the Form Factors Equations

How to solve these equations?

We can use the factorization for writing the solution as

$$F_n^{\Phi}(\beta_1,\ldots,\beta_n) = K_n^{\Phi}(\beta_1,\ldots,\beta_n)\prod_{i<j}^{n} F_{\min}(\beta_i - \beta_j)$$

$$F_{ab}^{\min}(\beta) = S_{ab}(\beta)\, F_{ab}^{\min}(-\beta)$$

$$F_{ab}^{\min}(i\pi + \beta) = F_{ab}^{\min}(i\pi - \beta)$$

$F_{ab}^{\min}(\beta)$ does not have any pole or zeros in the physical strip

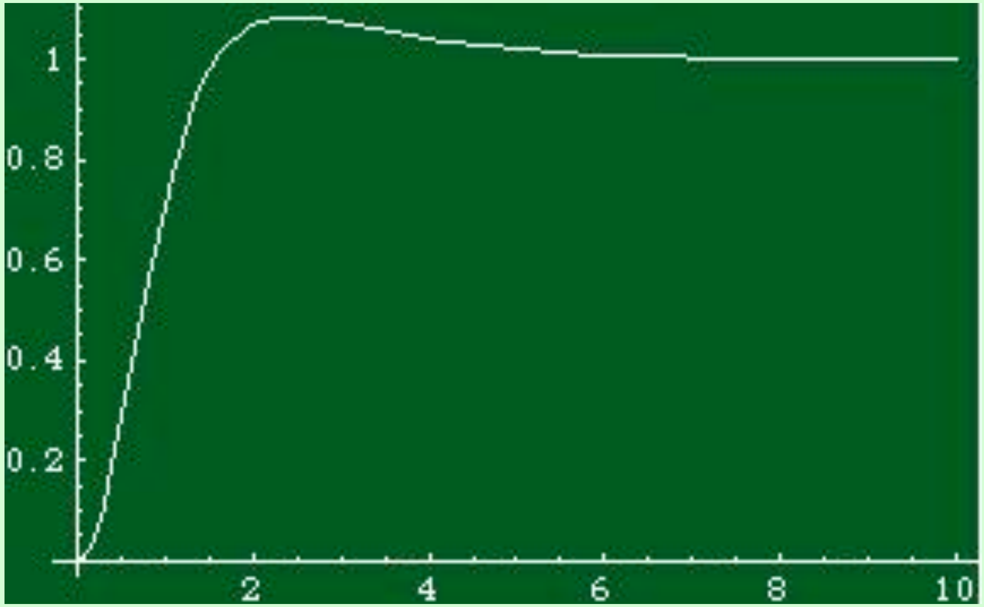
$$K_n^{\Phi}(.,\beta_i,\ldots,\beta_j,\ldots) = K_n^{\Phi}(.,\beta_j,\ldots,\beta_i,\ldots)$$

$$K_n^{\Phi}(\ldots,\beta_i + 2\pi i,\ldots,\beta_j,\ldots) = K_n^{\Phi}(.,\beta_i,\ldots,\beta_j,\ldots)$$

$K_n^{\Phi}\left(\left\{\beta_k\right\}\right)$ has the proper set of poles for solving the recursive eqs.

Minimal Form Factor of Sh-Gordon model

$$F_{\min}(\beta) = N \exp\left[4\int_0^\infty \frac{dt}{t} \frac{\sinh\frac{tB}{2}\sinh\frac{t(1-B)}{2}}{\sinh t \cosh\frac{t}{2}} \sin^2\left[\frac{(i\pi-\beta)t}{2\pi}\right]\right]$$



$$F_{\min}(\beta + i\pi) F_{\min}(\beta) = \frac{\sinh\beta}{\sinh\beta + \sin\pi B}$$

Infinite product representation

$$\int_0^\infty \frac{dt}{t} e^{-\beta t} \sin^2 at = \frac{1}{4} \log\left[1 + \left(\frac{2a}{\beta}\right)^2\right]$$

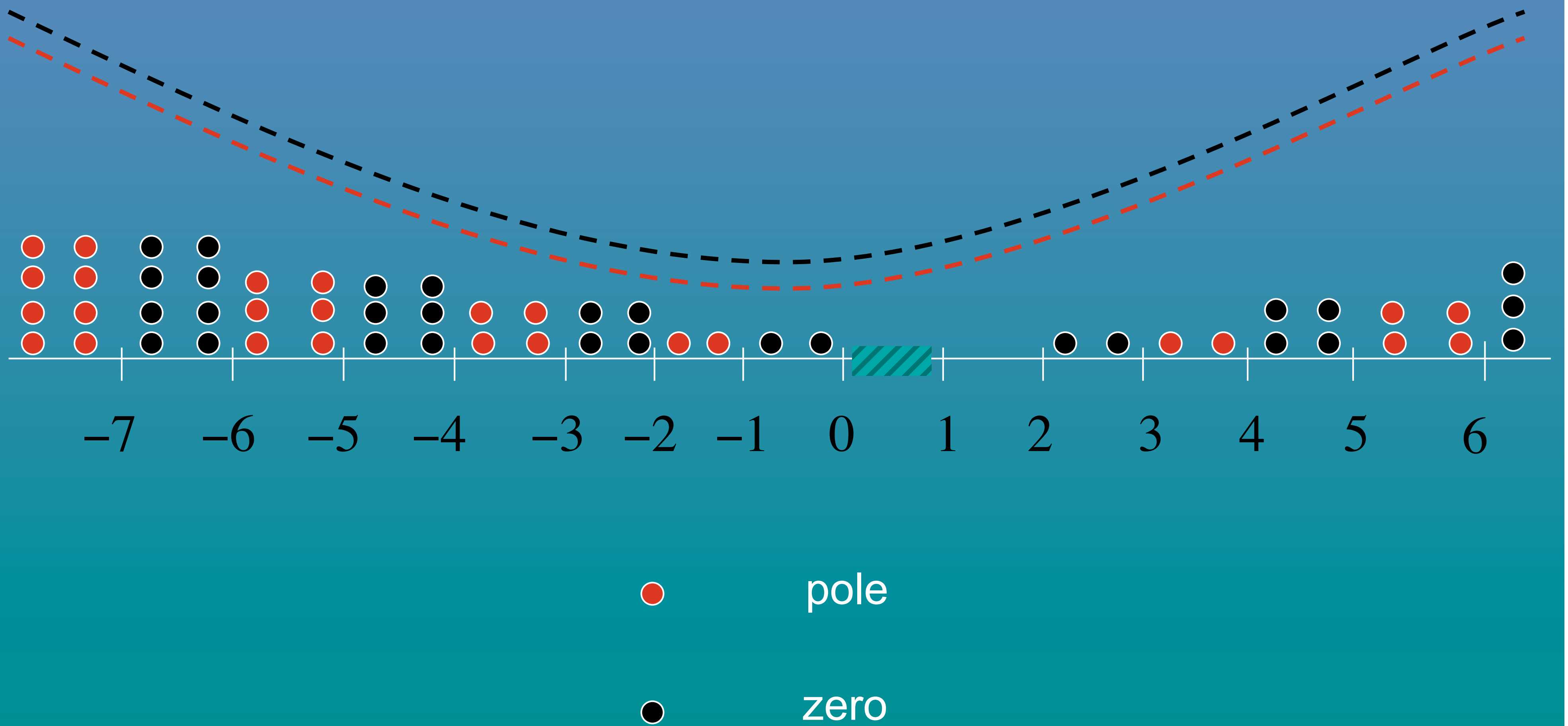
$$g_\alpha(\beta) = \exp\left[2\int_0^\infty \frac{dt}{t} \frac{\cosh \dfrac{t(1-2\alpha)}{2}}{\cosh \dfrac{t}{2} \sinh t} \sin^2\left[\hat{\beta} t\right]\right]$$

$$= \prod_{k=0}^{\infty} \left[\frac{\left(1 + \left(\dfrac{\hat{\beta}}{k+1-a/2}\right)^2\right)}{\left(1 + \left(\dfrac{\hat{\beta}}{k+1+a/2}\right)^2\right)} \frac{\left(1 + \left(\dfrac{\hat{\beta}}{k+1/2+a/2}\right)^2\right)}{\left(1 + \left(\dfrac{\hat{\beta}}{k+3/2-a/2}\right)^2\right)}\right]^{k+1}$$

Analytic structure

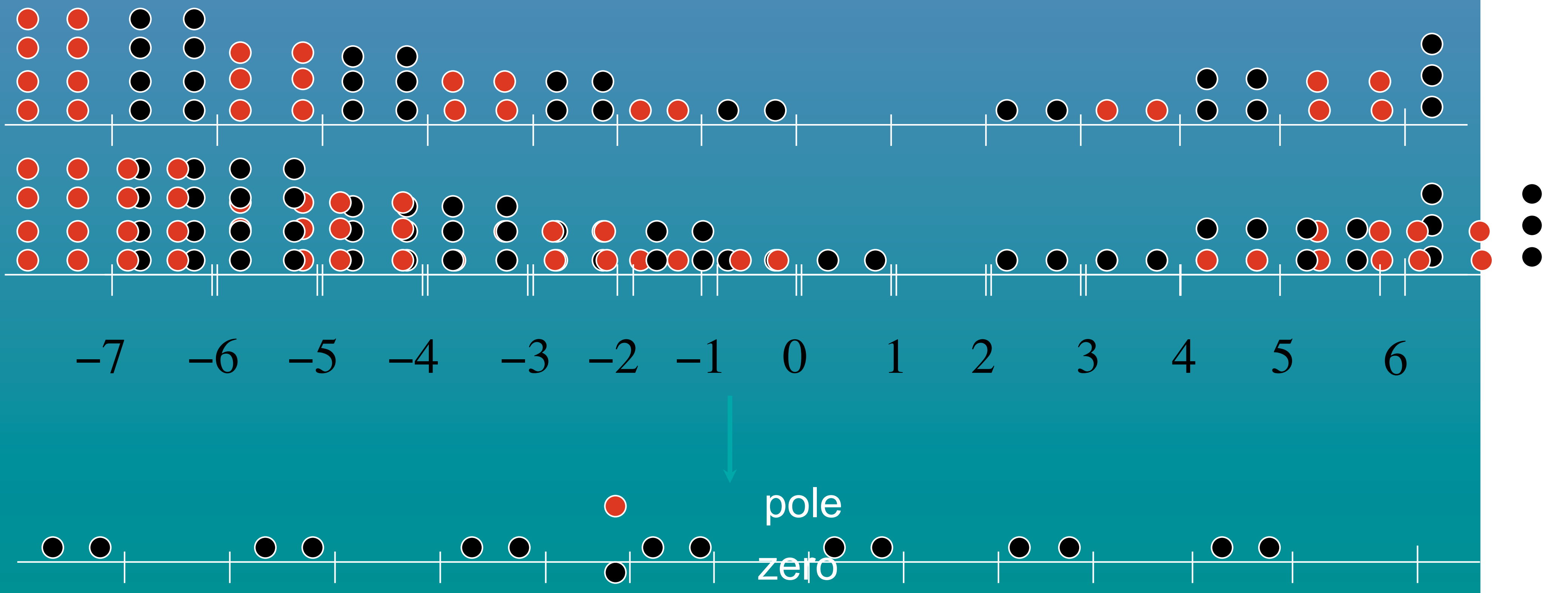
$g_a(\beta)$ has a sequence of increasing number of poles and zeros

along the immaginary axes, none of them in the physical strip



Functional equation

$$g_a(\beta + i\pi)\, g_a(\beta) = -i\,\frac{g_a(0)}{\sin \pi a}\left(\sinh \beta + i \sin \pi a\right)$$



Parameterization of the FF of scalar operators

$$F_n^{\Phi}(\beta_1,\dots,\beta_n) = H_n^{\Phi} Q_n^{\Phi}(x_1,\dots,x_n) \prod_{i<j}^{n} \frac{F_{\min}(\beta_{ij})}{x_i + x_j} \qquad ; \quad x_i = e^{\beta_i}$$

This symmetric polynomial contains all kinematical poles

This is also a symmetric polynomial, that contains the information on the operators Φ

$$\beta_k = \beta_j + i\pi$$

Its total degree is fixed by Lorentz invariance

$$d^t(Q_n^{\Phi}) = \frac{n(n-1)}{2}$$

Elementary symmetric polynomials

- Generating functions

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=0}^{n} x^{n-k}\sigma_k^{(n)}(x_1,\dots,x_n)$$

- Explicitly

$$\sigma_0^{(n)}(x_1,\dots,x_x) = 1$$

$$\sigma_1^{(n)}(x_1,\dots,x_x) = x_1 + x_2 + \cdots + x_n$$

$$\sigma_2^{(n)}(x_1,\dots,x_x) = x_1x_2 + x_1x_3 + \cdots$$

$$\cdots$$

$$\sigma_n^{(n)}(x_1,\dots,x_x) = x_1x_2x_3\cdots x_n$$

- The total degree of $\sigma_k^{(n)}$ is **k** and its partial degree is 1

Minimal Solutions (self-cluster operators)

(Koubek, GM)

$$\mathbb{Q}_n^{\Upsilon}[k] = \det M(k)$$

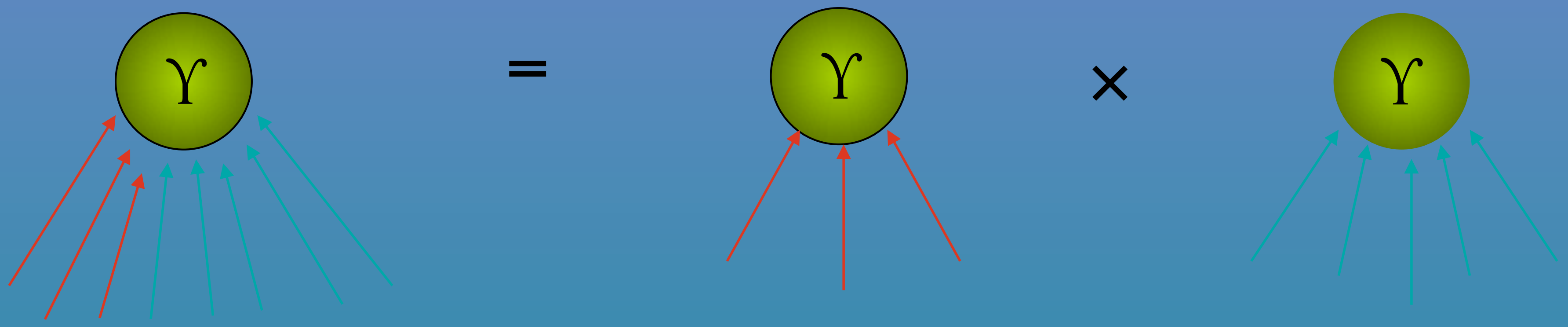
M(k) is an (n-1) x (n-1) matrix with entries

$$M_{ij}(k) = \sigma_{2j-i}^{(n)}\,[k+j-i]$$

$$M_{ij} = \begin{pmatrix} [k]\,\sigma_1 & [k+1]\,\sigma_3 & [k+2]\,\sigma_5 & \cdots \\ [k-1] & [k]\,\sigma_2 & [k+1]\,\sigma_4 & \cdots \\ 0 & [k-1]\,\sigma_1 & [k]\,\sigma_3 & \cdots \\ 0 & \cdots & \cdots & \cdots \end{pmatrix}$$

$$[k] \equiv \frac{\sin(k\pi B)}{\sin(\pi B)}$$

$$\lim_{\Delta\to\infty} F_n^{\Upsilon}(\beta_1+\Delta,\beta_2+\Delta,\ldots,\beta_m+\Delta;\beta_{m+1},\ldots,\beta_n) = F_m^{\Upsilon}(\beta_1,\ldots,\beta_m)F_{n-m}^{\Upsilon}(\beta_{m+1},\ldots,\beta_n)$$

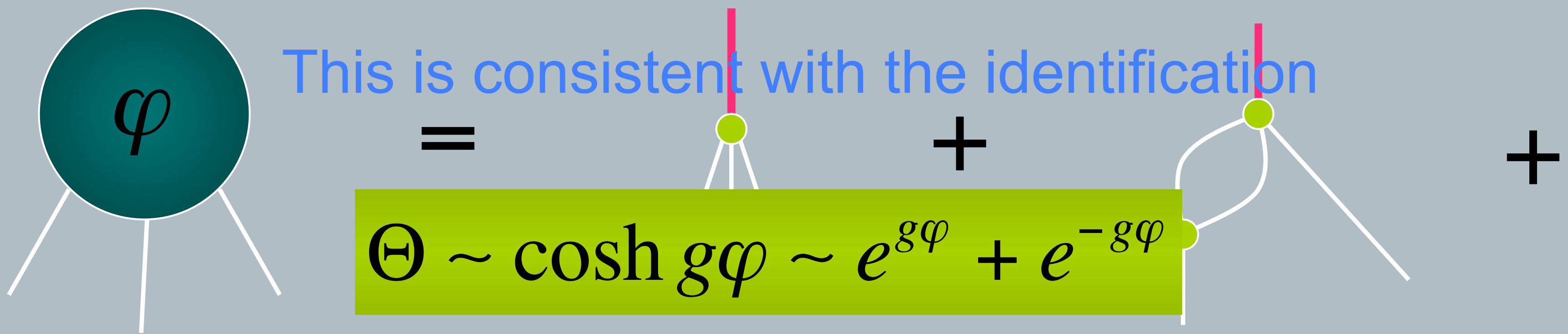


$$\Upsilon_k \longrightarrow e^{kg\varphi}$$

FF of basic operators

$$\varphi \longrightarrow Q_n^{\varphi} = \mathbb{Q}_n[k=0]$$

$$\Theta \longrightarrow Q_n^{\Theta} = \mathbb{Q}_n[k=1] + \mathbb{Q}_n[k=-1]$$



This is consistent with the identification

$$\Theta \sim \cosh g\varphi \sim e^{g\varphi} + e^{-g\varphi}$$

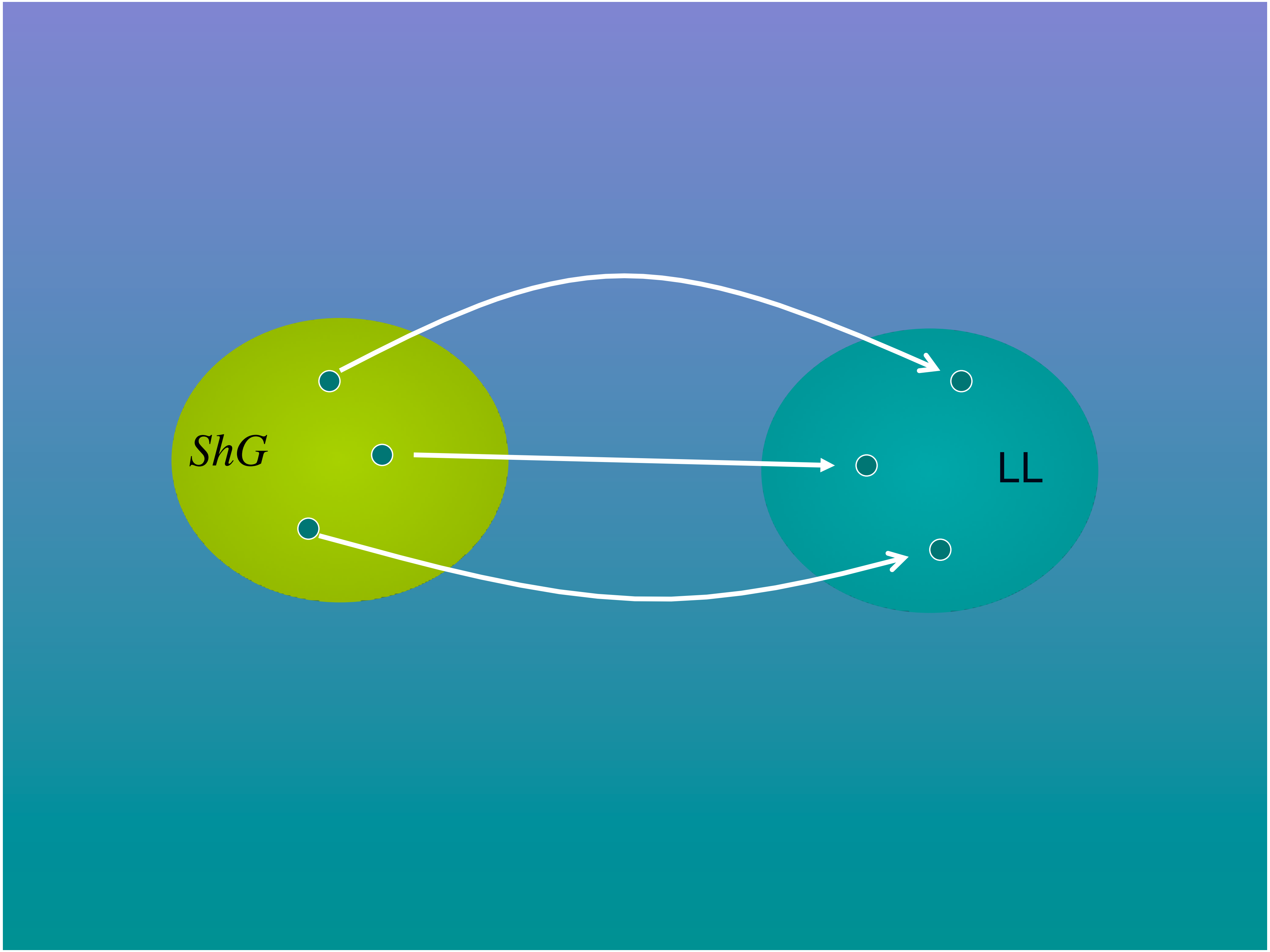
and with the c-theorem sum rule

Exact matrix elements of all other operators, such as

$$\Phi^{m}$$

are obtained by expanding O(m) the form factors of exponential operators

$$\langle :\phi^{2k}:\rangle \rightarrow \left(\frac{\hbar^2}{2m}\right)^k \binom{2k}{k} \langle \psi^{+k} \psi^{k} \rangle$$



ShG
LL

Finite T and finite density correlators

$$< A >_{T,n} = \frac{tr\left(e^{-\beta(H-\mu N)} A\right)}{tr\left(e^{-\beta(H-\mu N)}\right)}$$

In integrable theories there is a very efficient way of computing such quantity (LeClair-Mussardo formula)

$$< A >_{T,n} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_k}{2\pi} \left(\prod_{j=1}^{k} \frac{1}{1+e^{\varepsilon(\beta_j)}} \right)$$
$$< \beta_k, \ldots, \beta_1 \mid A(0,0) \mid \beta_1, \ldots, \beta_k >_{conn}$$

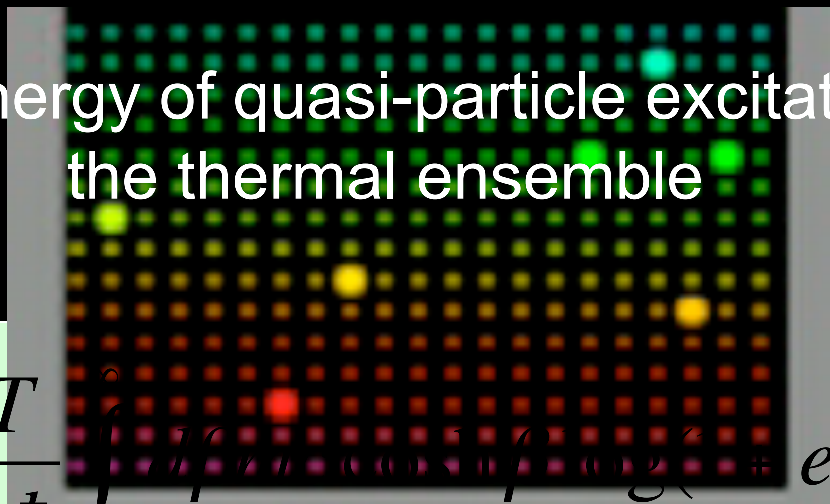
where $\varepsilon(\beta)$ is solution of the Thermodynamic Bethe Ansatz eqs

Thermodynamics Bethe Ansatz

$$\varepsilon(\beta) = \frac{mc^2}{kT}\cosh\beta - \frac{\mu}{kT} - \int_{-\infty}^{\infty}\frac{d\beta'}{2\pi}\varphi(\beta-\beta')\log(1+e^{-\varepsilon(\beta')})$$

$$\varphi(\beta) = -i\frac{\partial}{\partial\beta}\log S(\beta)$$

ε(β) is the energy of quasi-particle excitation above the thermal ensemble



$$F/L = -\frac{kT}{2\pi\hbar}\int_{-\infty}^{\infty} d\beta'\, mc\cosh\beta' \log(1+e^{-\varepsilon(\beta')}) + \mu N$$

Final expression of local correlators

$$< (\Psi^{+}\Psi)^{k} >_{T,n} = \left(\frac{\hbar^{2}}{2m}\right)^{-k} \sum_{m=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \frac{d\beta_{1}}{2\pi} \cdots \frac{d\beta_{n}}{2\pi} \left(\prod_{j=1}^{n} \frac{1}{1+e^{\varepsilon(\beta_{j})}} \right)$$

$$< \beta_{n},\ldots,\beta_{1} \,|\, :\Phi^{2k}: (0,0) \,|\, \beta_{1},\ldots,\beta_{n} >_{conn} \equiv n^{k} g_{k}(\gamma,\tau)$$

The series is fastly convergent!

- $g_1=1$
- g_2: Hellman-Feynman theorem
- g_3 known only at T=0 (Cheianov et al.)

Check. $g_1=1$

$$\langle \psi^+\psi \rangle = \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} f(p_1)\frac{1}{\hbar} + \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} f(p_1) \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} f(p_2)\frac{1}{\hbar}\tilde{\varphi}(p_1 - p_2) +$$

$$+\int_{-\infty}^{\infty} \frac{dp_1}{2\pi} f(p_1) \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} f(p_2) \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} f(p_3)\frac{1}{\hbar}\tilde{\varphi}(p_1 - p_2)\,\tilde{\varphi}(p_2 - p_3) + \ldots$$

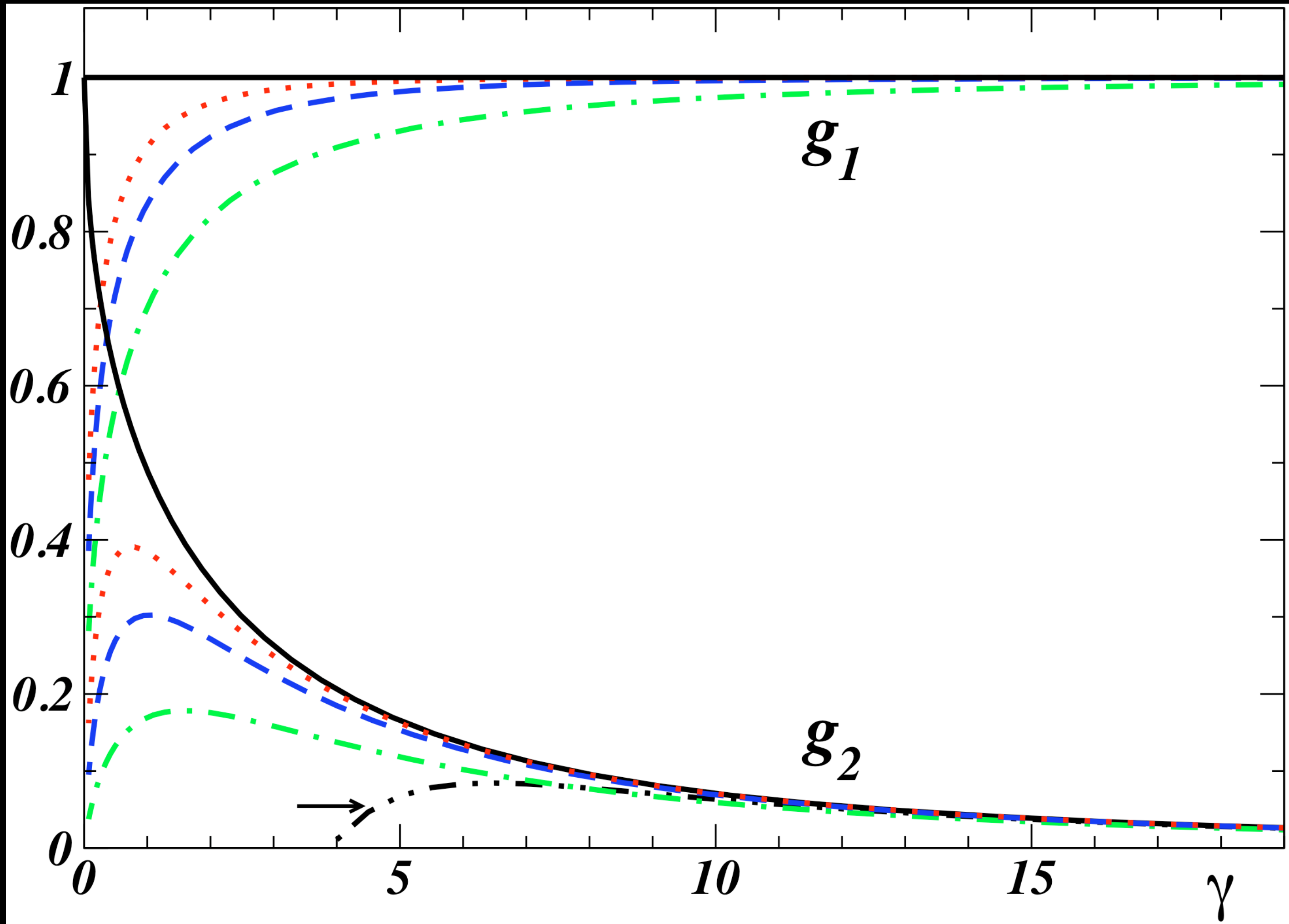
This can be resummed by the TBA equations

$$\tilde{\rho}(p) = \frac{1}{2\pi\hbar} + \int_{-\infty}^{\infty} \frac{dp'}{2\pi} f(p)\tilde{\varphi}(p - p')\tilde{\rho}(p') = n$$

$$\downarrow$$

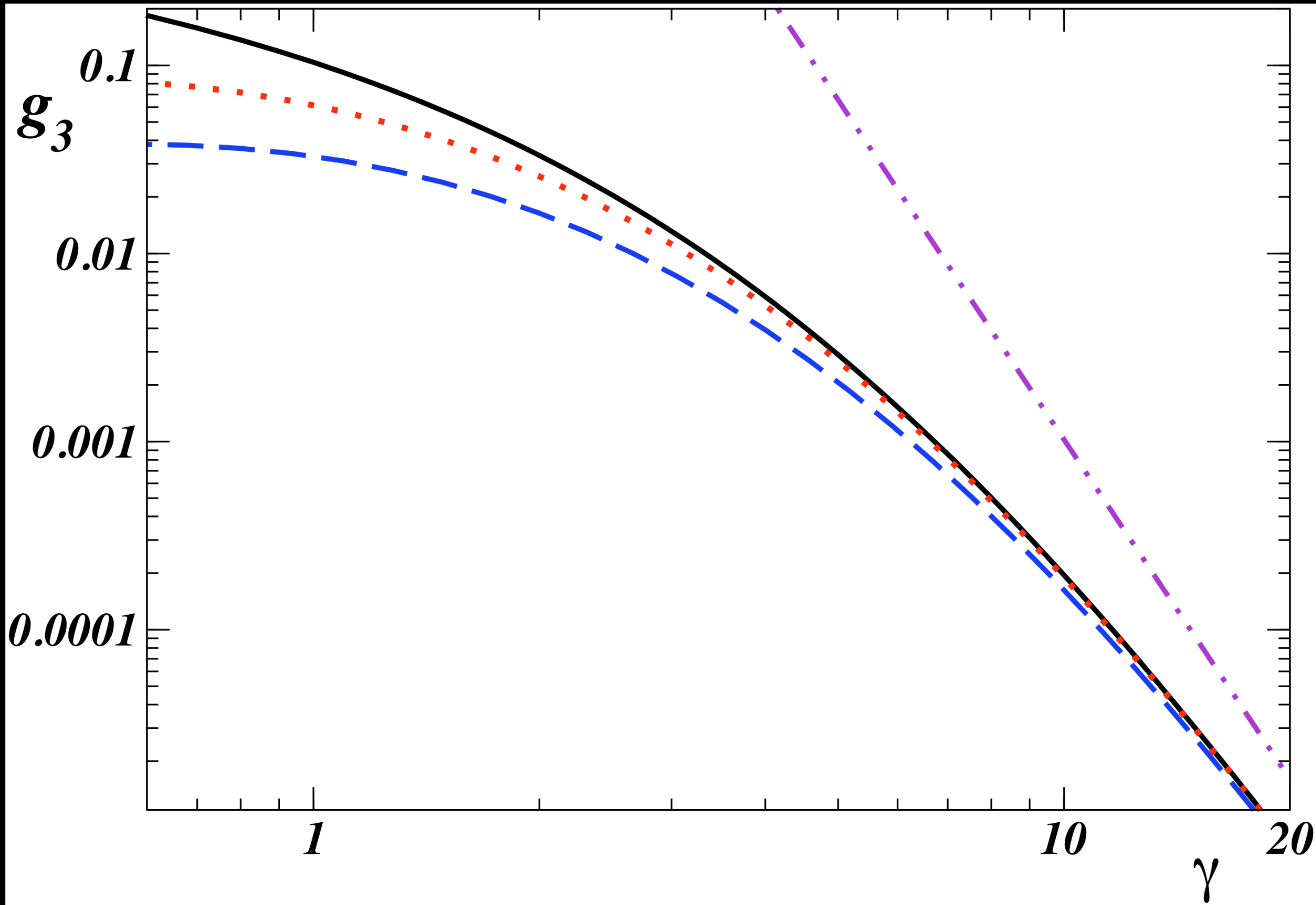
$$\langle \psi^+\psi \rangle = \int_{-\infty}^{\infty} dp\, f(p)\tilde{\rho}(p) = n$$

g_1 and g_2 at T=0



g1 and g2 at T = 0 using form factors up to n = 4, 6 and 8 particles, respectively with green dot-dashed, blue dashed and red dotted lines.

g_3 at T=0



g3 at T = 0 with form factors up to n = 6 and 8 particles. The exact value is given by the solid line whereas the purple dot-dashed line above corresponds to the leading order expression

Strong coupling, γ>>1, T=0

Leading term

$$g_k(\gamma) = \frac{k!}{2^k}\left(\frac{\pi}{\gamma}\right)^{k(k-1)} I_k + \ldots$$

$$I_k = \int_{-1}^{1} dx_1 \ldots \int_{-1}^{1} dx_k \prod_{i<j}^{k} (x_i - x_j)^2$$

in agreement with Gangardt at al.

But we can also compute next leading terms

- $$g_2 = \frac{4\pi^2}{3\gamma^2}\left(1 - \frac{6}{\gamma} - (24 - \frac{8\pi^2}{5})\frac{1}{\gamma^2}\right) + O(\gamma^{-5})$$
- $$g_3 = \frac{16\pi^6}{15\gamma^6}\left(1 - \frac{16}{\gamma}\right) + O(\gamma^{-8})$$

Asymptotic results for T>0

For $\gamma^2 >> \tau >> 1$

$$g_k(\gamma,\tau) = \left(\frac{\tau}{\gamma^2}\right)^{k(k-1)/2} J_k$$

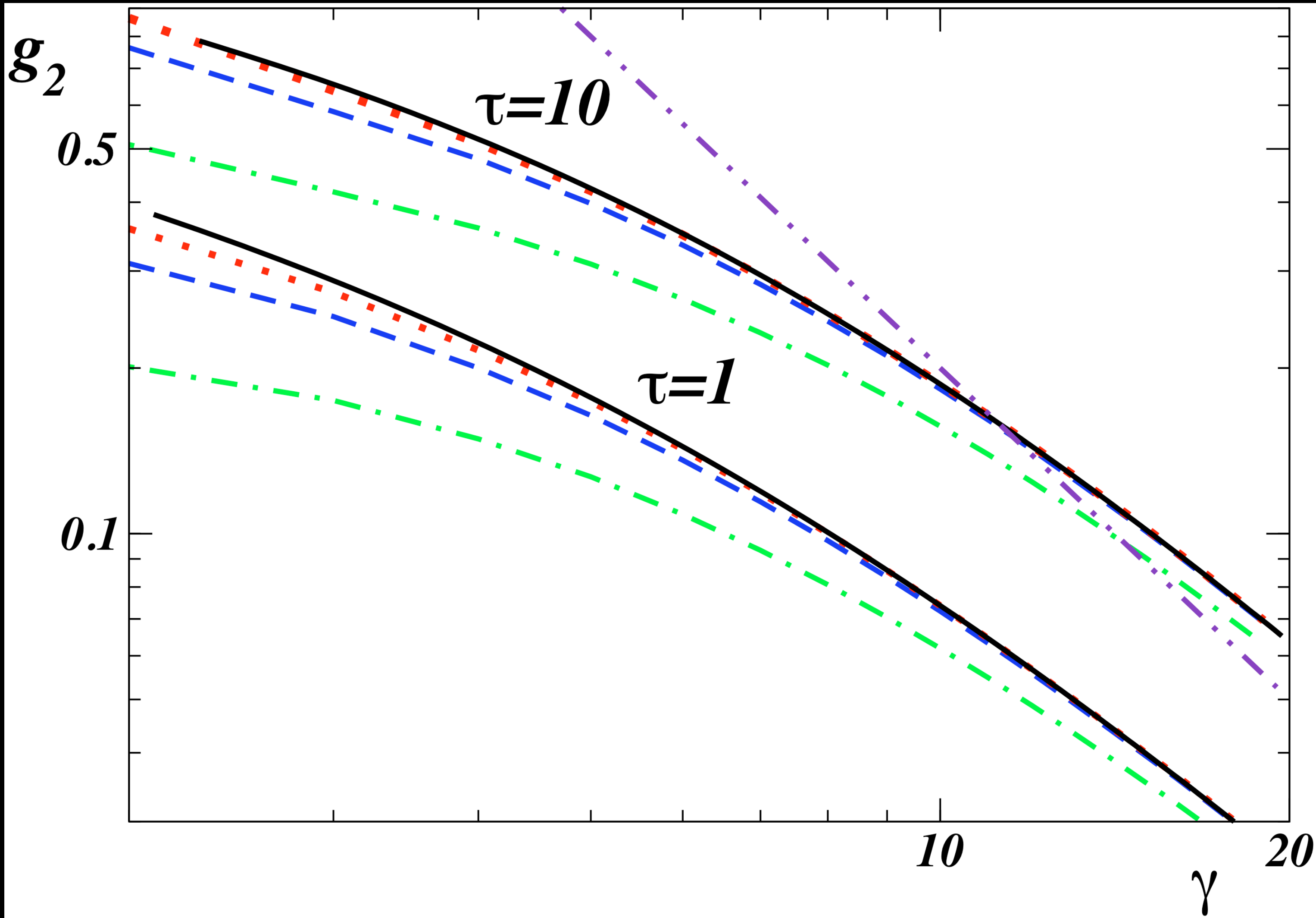
$$J_k = \frac{k!}{\pi^{k/2}} \int_{-1}^{1} dx_1 \ldots \int_{-1}^{1} dx_k \prod_{i<j}^{k} (x_i - x_j)^2 e^{-\sum_{i=1}^{k} x^2} = \frac{B_k}{2^{k(k-1)/2}}$$

once again, in agreement with Gangardt at al.

g_2 at T>0 from Hellman-Feynman theorem

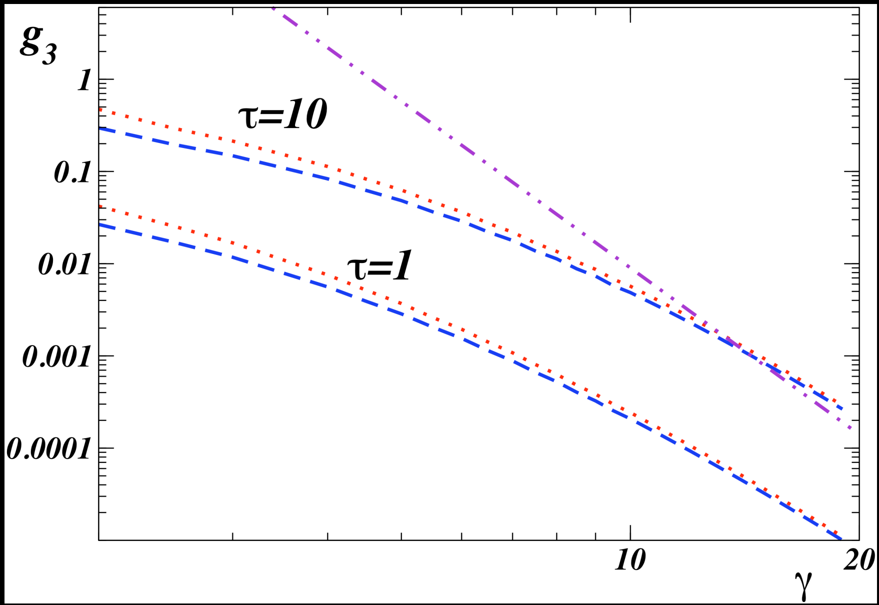
$$<\left(\Psi^{+}\Psi\right)^{2}>_{n,T} = \frac{d}{d\lambda}\left(\frac{F}{L}\right)$$

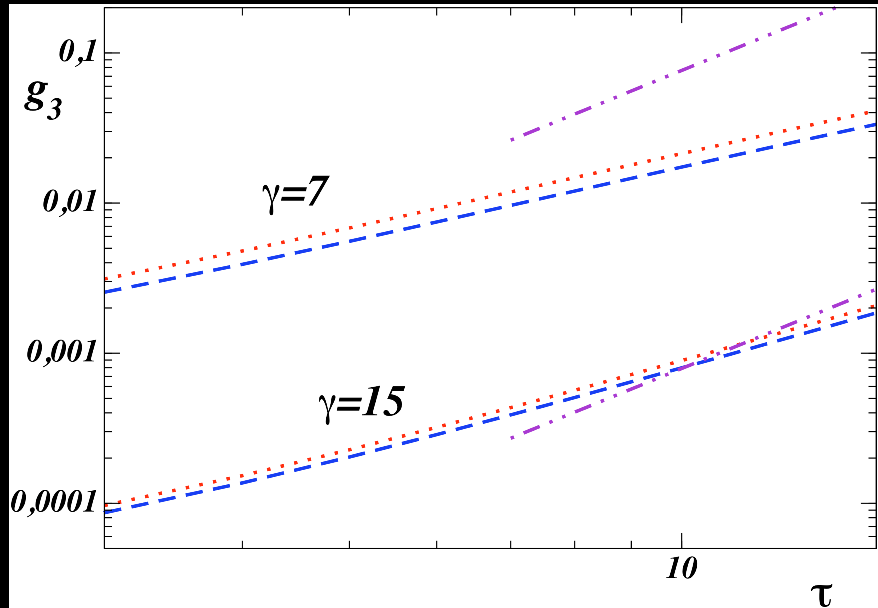
$$g_2(\gamma,\tau) = \tau\frac{d}{d\lambda}\left(\frac{\mu}{kT} - \gamma\int_{-\infty}^{\infty}\frac{dq}{2\pi}\log(1+e^{-\varepsilon(q)})\right)$$



g2 at τ = 1 and τ = 10 using form factors up to n = 4, 6 and 8 particles. The solid lines show the exact result, while the purple dot-dot-dashed line is the leading order expression.

g_3 at T>0





g_3 vs γ at τ= 1 and τ = 10

g_3 vs τ at γ= 7 and γ = 15

The blue dashed and the red dotted lines refer to n = 6 and 8 particles, respectively. The purple lines show the asymptotic results.

Conclusions

- Unified approach to compute correlation functions in LL model
- The method can be generalized also to compute correlators in highly excited states (Super-Tonks gases)
- The relative small value of g_3 explains the thermodynamic stability of the LL gas
- By product: long standing problem of computing matrix elements through Algebraic Bethe Ansatz can be drastically simplified