Tutorial on Scale and Conformal symmetries in diverse dimensions

R. Jackiw
MIT

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These days mathematical physicists are closely investigating scale and conformal transformations. Familiarity with theories that are invariant against these transformations — at least on the classical, prequantized level — extends for over a hundred year. Nevertheless there remain features, not unknown to some, that generally have fallen into obscurity. Therefore we take this occasion of a birthday meeting for Pasquale Sodano to bring into light some of these forgotten topics.

We shall describe the relation between scale and conformal transformations, and state conditions that must be satisfied by invariant theories. Thereby we expose dimensional peculiarities and universalities of scale and conformal transformations

Conformal Group of Transformations
(on a multi-component field Φ)

$$\text{translations} \; : \; \delta_T^\sigma \Phi(x) = \partial^\sigma \Phi(x)$$

$$\text{Lorentz rotations} \; : \; \delta_L^{\sigma\tau} \Phi(x) = (x^\sigma \partial^\tau - x^\tau \partial^\sigma + \Sigma^{\sigma\tau}) \; \Phi(x)$$

$$\text{dilation} \; : \; \delta_S \Phi(x) = (x^\tau \partial_\tau + d) \; \Phi(x)$$

$$d = \frac{D-2}{2} \qquad \text{(free bosons in D dimensions)}$$

$$\text{special conformal (primary field)} \; : \; \delta_C^\sigma \Phi(x) = (2x^\sigma x^\tau - g^{\sigma\tau} x^2) \, \partial_\tau \Phi(x) + 2x_\tau \, (g^{\tau\sigma} d - \Sigma^{\tau\sigma}) \, \Phi(x)$$

$\Rightarrow$ $\quad \delta x^\mu = -f^\mu(x)$

$f^\mu(x) = a^\mu, \quad \omega^{\mu\alpha} x_\alpha \quad (\omega^{\mu\alpha} = -\omega^{\alpha\mu}), \quad c x^\mu, \quad 2c_\alpha x^\alpha x^\mu - c^\mu x^2$

conformal Killing vector

$$\partial_\mu f_\nu + \partial_\nu f_\mu = \frac{2}{D} \, g_{\mu\nu} \, \partial_\alpha f^\alpha$$

Lorentz: $SO(D-1,1) \oplus$ Poincaré : $ISO(D-1,1)$,

$\oplus$Scale & Conformal : $SO(D,2)$

Relation between Scale and Conformal Symmetries

Conformal $\oplus$ Translation $\Rightarrow$ Lorentz $\oplus$ Dilation

$$[\delta_T^\sigma, \delta_C^\tau] = -2\, g^{\sigma\tau}\, \delta_S + 2\, \delta_L^{\sigma\tau}$$

Conditions for conformal symmetry:

1) Group theoretic: need scale symmetry

2) Dynamical: $\mathcal{L}\ (\partial_\mu \Phi, \Phi)$

$$V^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi}\ (g^{\mu\alpha}\, d - \Sigma^{\mu\alpha})\, \Phi$$

"Field virial" V^α must be total derivative: $V^\alpha = \partial_\beta\, \sigma^{\alpha\beta}$

$\Rightarrow$ energy momentum tenor $\theta^{\mu\nu}$ can be improved so that it is traceless

$$\theta^{\mu\nu} \to \theta^{\mu\nu}_{CCJ} \quad , \quad g_{\mu\nu}\, \theta^{\mu\nu}_{CCJ} = 0$$

$$\text{(Bessel-Hagen)} \quad J^\mu_f = \theta^{\mu\nu}_{CCJ}\, f_\nu \quad , \quad \partial_\mu J^\mu_f = 0$$

NB: It is possible to have scale symmetry without conformal symmetry when field virial is not a total derivation

$\Rightarrow$ no traceless energy-momentum tensor exists, scale current is not of Bessel-Hagen form, involves terms beyond energy-momentum tensor

It happens that in many models scale symmetry is broken (*e.g.* by mass terms) but field virial is total derivative $\Rightarrow$ obstacle to conformal invariance is scale non-invariance. "scale symmetry implies conformal symmetry" NOT generally true.

Scale Symmetric but Conformally un-Symmetric Models

(A) $\mathcal{L}(\partial_\mu \varphi, \varphi) = \mathsf{L}\left(\frac{\partial_\mu \varphi \, \partial^\mu \varphi}{\varphi^{\frac{2D}{D-2}}}\right) \varphi^{\frac{2D}{D-2}}$

Scale invariant with any $L(z)$
Conformally invariant only with $L(z) = L_0 + L_1 z$

$$\Rightarrow \; \mathcal{L} = \frac{1}{2}\, \partial_\mu \varphi \, \partial^\mu \varphi - \lambda \, \varphi^{\frac{2D}{D-2}}$$

(B) Free Maxwell theory (vector potential based)

$$\mathcal{L} = -\frac{1}{4}\, F^{\alpha\beta} \, F_{\alpha\beta}$$

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$\theta^{\mu\nu} = -F^{\mu\alpha} \, F^{\nu}{}_{\alpha} + \frac{g^{\mu\nu}}{4}\, F^{\alpha\beta} \, F_{\alpha\beta}$$

$$\theta^{\mu}{}_{\mu} = (-1 + D/4)\; F^{\alpha\beta} \, F_{\alpha\beta} \qquad \neq 0 \;\; D \neq 4$$

(a) Scale Symmetry

$$\delta_S \, A_\alpha(x) = (x^\tau \, \partial_\tau + \frac{D-2}{2})\; A_\alpha(x)$$

$$\delta_S F_{\alpha\beta}(x) = (x^\tau \, \partial_\tau + \frac{D}{2}) F_{\alpha\beta}(x)$$

$$J^\mu_S(x) = \theta^{\mu}{}_{\alpha}(x) \, x^\alpha + \underbrace{\frac{4-D}{2} \, F^{\mu\alpha}(x) \, A_\alpha(x)}_{\text{field virial}}$$

Summary

$$
\begin{aligned}
J^\mu_f &= \theta^{\mu\alpha} f_\alpha + \frac{4-D}{2D}\, \partial_\alpha f^\alpha F^{\mu\beta} A_\beta \\
\partial_\mu J^\mu_f &= \frac{4-D}{2D}\, \partial_\mu \partial_\alpha f^\alpha F^{\mu\beta} A_\beta \\
\partial_\mu \partial_\alpha f^\alpha &= 0 \quad \text{except conformal}
\end{aligned}
$$

Final observation

$$
\delta_f A_\alpha \neq \mathsf{L}_f A_\alpha \quad (D \neq 4)
$$

$\uparrow$ Lie derivative

$$
\begin{aligned}
\mathsf{L}_f A_\alpha &= f^\mu \partial_\mu A_\alpha + \partial_\alpha f^\mu A_\mu \\
&= f^\mu F_{\mu\alpha} + \partial_\alpha (f^\mu A_\mu) \\
\delta_f A_\alpha &= \mathsf{L}_f A_\alpha + \frac{D-4}{2D}\, \partial_\mu f^\mu A_\alpha
\end{aligned}
$$

NB J_S^μ is gauge variant, but charge

$$\int d^{D-1}x \ J_S^0 \quad \text{is not}$$

(apart from surface)

(b) Conformal Symmetry

$$\begin{aligned}
\delta_C^\sigma A_\alpha(x) &= \text{primary field} \\
\delta_C^\sigma F_{\alpha\beta}(x) &= \text{not primary field} \\
&= \triangle^\sigma F_{\alpha\beta} + (D-4)(g_\alpha^\sigma A_\beta - g_\beta^\sigma A_\alpha) \\
&\quad \uparrow\!__ \ \text{primary field} \\
\delta_C^\sigma \mathcal{L} &= \partial_\mu \ \left[(2x^\sigma x^\mu - g^{\sigma\mu} x^2)\mathcal{L})\right] + (4-D) \ F^{\sigma\tau} A_\tau \\
\text{(recall) } V^\alpha &= \frac{4-D}{2} F^{\alpha\beta} A_\beta \neq \partial_\beta \sigma^{\alpha\beta}
\end{aligned}$$

No conformal symmetry

(based on vector potential)

(C) Free 3-D Maxwell Theory (scalar potential based)

$$F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\,\partial^\gamma\,\varphi$$

Equation of Motion: $\partial^\alpha F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\,\partial^\alpha\,\partial^\gamma\,\varphi = 0$ (identity)

Bianchi Identity : $\frac{1}{2}\,\partial_\gamma\,\varepsilon^{\gamma\alpha\beta}\,F_{\alpha\beta} = \Box\,\varphi = 0$ (equation of motion)

$$\theta^{\mu\nu}_{CCJ} = \partial^\mu\,\varphi\,\partial^\nu\,\varphi - \frac{g^{\mu\nu}}{2}\,\partial_\alpha\,\varphi\,\partial^\alpha\,\varphi + \frac{1}{8}\,(g^{\mu\nu}\,\Box - \partial^\mu\,\partial^\nu)\,\varphi^2$$

$$g^{\mu\nu}\,\theta^{\mu\nu}_{CCJ} = 0$$

$$\delta_S\,\varphi(x) = \left(x^\tau\,\partial_\tau + \frac{1}{2}\right)\,\varphi(x)$$

$\delta^\sigma_C\,\varphi(x) =$ primary field

$\delta^\sigma_C\,\partial^\gamma\,\varphi =$ not primary field

$$\delta^\sigma_C\,F_{\alpha\beta}(x) = \triangle^\sigma\,F_{\alpha\beta}(x) + \varepsilon_{\alpha\beta}{}^{\sigma}\,\varphi(x)$$

↖not primary

$$\bar{\delta}^\sigma_C\,\mathcal{L} = \partial_\mu\left[(2x^\sigma\,x^\mu - g^{\sigma\mu}\,x^2)\,\mathcal{L} - \frac{1}{2}\,g^{\sigma\mu}\,\varphi^2\right]$$

$$\theta^{\mu\nu}{}_{CCJ} = -\frac{3}{4}\,F^{\mu\alpha}\,F^\nu{}_\alpha + \frac{1}{4}\,g^{\mu\nu}\,F^{\alpha_\beta}\,F_{\alpha_\beta} - \frac{\varphi}{16}\,\left(\partial^\mu\,\varepsilon^{\nu\alpha\beta}\,F_{\alpha\beta} + \partial^\nu\,\varepsilon^{\mu\alpha\beta}\,F^{\alpha_\beta}\right)$$

$$J^\mu_f = \theta^{\mu\nu}{}_{CCJ}\,f_\nu, \qquad \partial_\mu\,J^\mu_f = 0$$

Dimensional Ladder for Scalar N-component field Φ

In D dimensions, N component field $\mathbf{\Phi}$

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \mathbf{\Phi} \cdot \partial^\mu \mathbf{\Phi} - \lambda \, (\mathbf{\Phi} \cdot \mathbf{\Phi})^{\frac{D}{D-2}}$$

is scale and conformally invariant

$$\begin{aligned} \delta_S \, \mathbf{\Phi}(x) &= \left(x^\tau \, \delta_\tau + \frac{D-2}{2} \right) \, \mathbf{\Phi}(x) \\ \delta_C^\sigma \, \mathbf{\Phi}(x) &= \left(2x^\sigma \, x^\tau - g^{\sigma\mu} \, x^2 \right) \, \partial_\tau \, \mathbf{\Phi}(x) + (D-2) \, x^\sigma \, \mathbf{\Phi}(x) \end{aligned}$$

Poincaré: $ISO(D-1,1)$, Scale & Conformal: $SO(D,2)$

Continue to $D=1$ (time): $\mathbf{\Phi}(t,\mathbf{x}) \rightarrow \mathbf{q}(t)$

$$\mathcal{L} \rightarrow L = \frac{1}{2}\, \partial_t \; \mathbf{q} \cdot \partial_t \, \mathbf{q} - \lambda / q^2$$

Poincaré $\rightarrow$ time translation $\delta_t \, \mathbf{q} = \partial_t \, \mathbf{q}$

Scale $\rightarrow \delta_S \, \mathbf{q} = \left(t \, \partial_t - \frac{1}{2} \right) \; \mathbf{q}$

Conformal $\rightarrow \delta_C \, \mathbf{q} = \left(t^2 \, \partial_t - t \right) \; \mathbf{q}$

Non relativistic conformal group $SO(D,2) \rightarrow SO(1,2)$