Non-Abelian vortices and lumps; their moduli spaces and substructures

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Theories of fundamental interactions
Perugia
June 25, 2010

In collaboration with: Roberto Auzzi, Minoru Eto, Toshiaki Fujimori, Yunguo Jiang, Kenichi Konishi, Takayuki Nagashima, Muneto Nitta, Keisuke Ohashi and Walter Vinci.

arXiv:1005.0557  SBG
arXiv:1002.0850  Konishi,SBG
arXiv:0906.0021  SBG
arXiv:0905.3540  Eto,Fujimori,SBG,Konishi,Nagashima,Nitta,Ohashi,Vinci
arXiv:0903.4471  Eto,Fujimori,SBG,Konishi,Nagashima,Nitta,Ohashi,Vinci
arXiv:0809.2014  Eto, Fujimori, SBG, Nitta, Ohashi
Plan of the talk

Introductory part

→ ○ motivation
  ● crash course on vortices
  ● summary of recent results in the field

Results

● $SO, USp$ theories
● $NL\sigma M$ lumps
● fractional vortices
● Chern-Simons vortices
Motivation

- Confinement of quarks in Yang-Mills theory (’t Hooft-Mandelstam)
- Electric-Magnetic duality (Dirac, Montonen-Olive)
- Non-Abelian monopoles and GNOW-duality (Goddard-Nuyts-Olive-Weinberg)
- Condensed matter systems – quantized vortices
- Fractional quantum Hall effect – Chern-Simons theory
- Neutron stars
- Cosmic strings
- etc.
“Dynamical Abelianization”? 

- Softly broken $\mathcal{N} = 2$ pure $SU(2)$ in Seiberg-Witten theory  
  $\rightarrow$ Abelian

- Generic $\mathcal{N} = 2$ super-Yang-Mills (SYM) with quarks  
  $\rightarrow$ non-Abelian

Hanany-Oz 1996, Carlino-Murayama-Konishi 2000

Question: magnetic monopoles of QCD are of Abelian or non-Abelian type?
Non-Abelian monopoles

Symmetry-breaking pattern:

\[ G \rightarrow H \neq U(1) \quad (1) \]

Field strength tensor:

\[ F_{ij} \sim \epsilon_{ijk} \frac{x_k}{r^3} (\beta \cdot T_{\text{Cartan}}) \]

Charge quantization:

\[ 2\beta \cdot \alpha = \mathbb{Z} \quad (2) \]

\( \alpha \) is the root vector of \( H \). Solution: \( \beta \) is any weight vector of the dual group \( \tilde{H} \) which has root vectors

\[ \alpha^* = \frac{\alpha}{\alpha \cdot \alpha} \quad (3) \]

The GNOW conjecture:

non-Abelian monopoles form multiplets of the dual group \( \tilde{H} \).

<table>
<thead>
<tr>
<th>( H )</th>
<th>( U(N) )</th>
<th>( SU(N) )</th>
<th>( SO(2N) )</th>
<th>( SO(2N + 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{H} )</td>
<td>( U(N) )</td>
<td>( SU(N) / \mathbb{Z}_N )</td>
<td>( Spin(2N) )</td>
<td>( USp(2N) )</td>
</tr>
</tbody>
</table>
Unfortunately, there are some obstacles:

1. topological obstruction
2. non-normalizable zero-modes – going to zero as $r^{-\frac{1}{2}}$

$\Rightarrow$ we cannot quantize the non-Abelian monopole

The transformations in $H$ and $\tilde{H}$, respectively are non-local. Problematic: Coulomb phase and (dual) Higgs phase.

Strategy:

$H_{\text{Higgs phase}} \Leftrightarrow \tilde{H}_{\text{confinement phase}}$
Benchmark model

\[ G \xrightarrow{\Lambda} H \xrightarrow{\mu} \mathbb{1} , \quad \Lambda \gg \mu . \quad (4) \]

Exact homotopy sequence:

\[ \ldots \rightarrow \pi_2 (G) \rightarrow \pi_2 \left( \frac{G}{H} \right) \rightarrow \pi_1 (H) \rightarrow \pi_1 (G) \rightarrow \ldots \quad (5) \]

Example:

\[ 1 = \pi_2 (SU(3)) \rightarrow \pi_2 \left( \frac{SU(3)}{SU(2) \times U(1)} \right) \rightarrow \pi_1 (SU(2) \times U(1)) \rightarrow \pi_1 (SU(3)) = 1 . \quad (6) \]
Paradox?

Complete system

\[ G \rightarrow 1 \] \hspace{1cm} (7)

\[ \pi_1(G) = 1, \] \hspace{1cm} (8)

\[ \Rightarrow \text{no monopoles} \]

\[ \Rightarrow \text{no vortices} \]

Solution:

Vortices confine the monopoles

Idea: vortex transformation = monopole transformation

Auzzi-Bolognesi-Evslin-Konishi 2004

Auzzi-Bolognesi-Evslin-Konishi-Yung 2003
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Abrikosov-Nielsen-Olesen vortex

\[ \pi_1(S^1) = \mathbb{Z} \ni k : \text{winding number} = \text{vorticity} \]

\(\xi\) : Fayet-Iliopoulos parameter – theory on the Higgs branch
Derrick’s theorem

No finite energy scalar field configuration in more than one spatial dimension, other than the vacuum can have a stationary point

- $d = 1$: domain wall – OK
- $d = 2$: vortex – stabilized by
  - flux
  - $V = 0 \Rightarrow$ sigma model lumps – harmonic maps
  - $J \neq 0$ – $Q$-lumps
The vortex stabilized by magnetic flux

Competition of forces:

- scalar field attractive force
- magnetic field repulsive force

\[ \beta \equiv \frac{m_{\text{Higgs}}}{m_\gamma} \] classifies the vortices into:

- \( \beta < 1 \): type I – not experimentally observable as the flux attracts and breaks the superconducting phase
- \( \beta > 1 \): type II – Abrikosov lattice
- \( \beta = 1 \): BPS – supersymmetry preserving
FIG. 2. Triangular vortex lattice at magnetic field 200 Oe in MgB$_2$ single crystal. Inset: FFT pattern in an arbitrary scale.
Non-Abelian embedding

\( U(N) \) theory with \( N_F = N \) flavors:

\[
q = \begin{pmatrix}
q_{11} & q_{12} & \cdots & q_{1N} \\
q_{21} & q_{22} & q_{2N} \\
\vdots & \cdots & \vdots \\
q_{N1} & \cdots & \cdots & q_{NN}
\end{pmatrix}, \quad \langle q \rangle = 1_N
\]  \hspace{1cm} (9)

Transformation

\[
q \rightarrow U_{\text{color}} q U_{\text{flavor}}^\dagger,
\]  \hspace{1cm} (10)

\( U_{\text{flavor}} = U_{\text{color}} \) global symmetry.

Embedding of ANO in \( U(N) \) theory:

\[
q^{\text{ANO}} = \begin{pmatrix}
q_{\text{ANO}} & 0 & \cdots & 0 \\
0 & \sqrt{\xi} & 0 \\
\vdots & \cdots & \vdots \\
0 & \cdots & \cdots & \sqrt{\xi}
\end{pmatrix}, \quad A_i^{\text{ANO}} = \begin{pmatrix}
A_i^{\text{ANO}} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix}.
\]
A whole family of solutions appears!
Non-Abelian moduli

Color-flavor rotation:

\[ q = U \begin{pmatrix} q^{\text{ANO}} & 0 & \cdots & 0 \\ 0 & \sqrt{\xi} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sqrt{\xi} \end{pmatrix} U^\dagger, \quad (11) \]

\[ A_i = U \begin{pmatrix} A_i^{\text{ANO}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} U^\dagger. \]

\[ U \in G_{\text{color+flavor}} \text{ global symmetry.} \]

Non-Abelian moduli parametrize:

\[ \frac{SU(N)}{SU(N - 1) \times U(1)} \simeq \mathbb{C}P^{N-1}. \quad (12) \]

\[ \Rightarrow \text{effective world-sheet symmetry.} \]
Moduli space the non-Abelian vortex

$U(N)$ theory:

$$\mathcal{M}_k = \left( \mathbb{C} \times \mathbb{C} \mathbb{P}^{N-1} \right)^k / \mathfrak{S}_k$$

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Summary of recent results

- A duality between 4 dim SYM and a 2 dim non-linear $\sigma$ model, realized by the non-Abelian vortex
  
  Hanany-Tong 2003, Shifman-Yung 2004

- $SO(5) \rightarrow U(2) \rightarrow 1$: regular monopoles are confined by $k = 2$ vortices.

  $k = 2$ moduli space of $U(2)$ vortices:

  $\mathbb{C}P^1 \times \mathbb{C}P^1 \rightarrow W\mathbb{C}P^1_{2,1,1}$


The vortex-monopole system transforms under the $3 + 1$ representation of $SU(2)$.

Group theory of vortices: single vortex $\Box : k$ vortices $\prod_{i=1}^{k} \Box$.

- The full moduli space:
  - In preparation
  - Eto-Isozumi-Nitta-Ohashi-Sakai 2005

- Reconnection of cosmic strings

- D-brane solitons in field theory, e.g. an instanton-monopole-vortex-domain wall system

- A Seiberg-like duality of non-Abelian semi-local vortices
• Non-BPS non-Abelian vortex interactions: distance-dependent forces (type I/I* and type II/II*)
  
Auzzi-Eto-Vinci 2007

• Non-Abelian vortices on a torus:

Lozano-Marques-Schaposnik 2007

• Non-Abelian vortices in dense QCD:


• A model of non-Abelian vortices without dynamical Abelianization

Dorigoni-Konishi-Ohashi 2008

• The stability of non-Abelian semi-local vortices

Auzzi-Eto-SBG-Konishi-Vinci 2008
• Multi-layer structure of non-Abelian vortices: Eto-Fujimori-Nitta-Ohashi-Sakai 2009

• Non-Abelian global vortices: Eto-Nakano-Nitta 2009

• Vortex description of quantum Hall ferromagnets Kimura 2009

• Non-Abelian vortex-string dynamics from non-linear realization: Liu-Nitta 2009


• Quantum Phases of a vortex string in $\mathcal{N} = 1^*$: Auzzi-Kumar 2009
• Non-Abelian Chern-Simons vortices with generic gauge groups:

• Moduli space metric for non-Abelian vortices
  – Compact Riemann surfaces:
  
  – Well-separated vortices:

• Low-energy $U(1) \times USp(2M)$ gauge theory from simple high-energy gauge group:
• D-branes in rotating phase-separated two-component Bose-Einstein condensates
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The model – general gauge group

$H$ theory: $\mathcal{N} = 2$ (8 supercharges), $U(1) \times G'$ super-Yang-Mills (4 dimensions) with $N_F$ massless hyper multiplets.

vector multiplet : $\{V, \Phi\}$

hyper multiplets : $\{Q, \tilde{Q}\}$

**Truncated** model, formally $\mathcal{N} = 1$, $U(1) \times G'$ SYM with $N_F$ quarks $Q$.

$$\mathcal{L} = \text{Tr} \left[ \int d^4 \theta \left( Q^\dagger e^{-V} Q + \xi V \right) + \frac{1}{g^2} \int d^2 \theta \left( WW + \text{h.c.} \right) \right],$$

$\xi > 0$ FI parameter.

$$\mathcal{L} = - \frac{1}{4e^2} F_{\mu\nu}^0 F_{\mu\nu}^0 - \frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a + \text{Tr} D_\mu H (D^\mu H)^\dagger$$

$$- \frac{e^2}{2} \left| \text{Tr} HH^\dagger t^0 - \frac{v^2}{\sqrt{2N}} \right|^2 - \frac{g^2}{2} \left| \text{Tr} HH^\dagger t^a \right|^2, \quad (13)$$

t$^a$ generators of $G'$. 
BPS equations

Bogomol’nyi bound:

\[ T \geq -\frac{v^2}{\sqrt{2N}} \int_C F_{12}^0 = 2\pi v^2 \nu , \]  

(14)
saturated by BPS-equations

\[ \bar{D}H = 0 , \quad F_{12}^0 = e^2 \left( \text{Tr} \ H H^\dagger t^0 - \frac{v^2}{\sqrt{2N}} \right) , \]  

(15)

\[ F_{12}^a = g^2 \text{Tr} \ H H^\dagger t^a , \]  

(16)
\[ \nu \text{ topological charge.} \]
Moduli matrix

Solution: holomorphic matrix + complexified gauge transformations:

\[ H = S^{-1}(z, \bar{z}) H_0(z) , \]  

(17)

with \( z = x^1 + ix^2 \), holomorphic \( N \times N_F \) matrix: moduli matrix, encodes all the moduli.

\[ S = sS' , \quad s \in \mathbb{C}^* , \quad S' \in G'^{\mathbb{C}} . \]  

(18)

\[ \bar{D}H = 0 , \quad \Rightarrow \quad \bar{A} = -iS^{-1}\bar{\partial}S . \]  

(19)

Residual symmetry – \( V \)-equivalence:

\[ (S, H_0) \sim V(z) (S, H_0) . \]  

(20)
Master equations

\[ \tilde{D}H = 0 \text{ solved. } \Omega = \omega \Omega' = SS'^\dagger, \ \Omega_0 = H_0(z)H_0^\dagger(z): \]

For \( SU(N) \)

\[ \bar{\partial} \partial \log \omega = -\frac{e^2}{4N} \left[ \frac{1}{\omega} \text{Tr} \Omega_0 \Omega'^{-1} - v^2 \right] , \quad (21) \]

\[ \bar{\partial} \left( \Omega' \partial \Omega'^{-1} \right) = \frac{g^2}{4\omega} \left[ \Omega_0 \Omega'^{-1} - \frac{1}{N} \text{Tr} \Omega_0 \Omega'^{-1} \right] , \quad (22) \]

\( SO, USp \):

\[ \bar{\partial} \left( \Omega' \partial \Omega'^{-1} \right) = \frac{g^2}{8\omega} \left[ \Omega_0 \Omega'^{-1} - J^\dagger \left( \Omega_0 \Omega'^{-1} \right)^T J \right] . \quad (23) \]

Existence and uniqueness is assumed.
Asymptotic behavior:
\[ s(z, \bar{z}) \sim |z|^{\nu} . \]  
(24)

Holomorphic \( G' \)-invariants:
\[ I_{G'}^i \left( H = s^{-1}S'^{-1}H_0 \right) = s^{-n_i}I_{G'}^i(H_0) , \]  
(25)

Boundary conditions:
\[ I_{G'}^i(H_0) \big|_{|z| \to \infty} = I_{\text{vev}}^i z^{\nu n_i} , \]  
(26)

Single valuedness condition
\[ \nu n_i \in \mathbb{Z}_+ , \]  
(27)

Solution:
\[ \nu = \frac{k}{n_0} , \quad k \in \mathbb{Z}_+ , \]  
(28)

\[ n_0 \equiv \gcd \{ n_i | I_{\text{vev}}^i \neq 0 \} . \]

\[ G = \frac{U(1) \times G'}{\mathbb{Z}_{n_0}} . \]  
(29)
$SU(N) = G'$

$SO, USp(2N) = G'$

$\frac{2\pi}{N}$

$\frac{2\pi}{2}$
A few groups

<table>
<thead>
<tr>
<th>$G$</th>
<th>$C_G$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(N)$</td>
<td>$\mathbb{Z}_N$</td>
<td>$k/N$</td>
</tr>
<tr>
<td>$SO(2N + 1)$</td>
<td>1</td>
<td>$k$</td>
</tr>
<tr>
<td>$USp(2N)$</td>
<td>$\mathbb{Z}_2$</td>
<td>$k/2$</td>
</tr>
<tr>
<td>$SO(2N)$</td>
<td>$\mathbb{Z}_2$</td>
<td>$k/2$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\mathbb{Z}_3$</td>
<td>$k/3$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\mathbb{Z}_2$</td>
<td>$k/2$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>1</td>
<td>$k$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1</td>
<td>$k$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>1</td>
<td>$k$</td>
</tr>
</tbody>
</table>
Examples

Holomorphic invariant conditions:

$G' = SU(N)$:

$$\det H_0(z) = z^k + O(z^{k-1}) ,$$

(30)

$G' = SO(2N), USp(2N)$: meson field:

$$M = H^T J H ,$$

(31)

Invariant tensor:

$$J_{SO(2N)} \equiv \sigma^1 \otimes 1_N , \quad J_{USp(2N)} \equiv i\sigma^2 \otimes 1_N ,$$

(32)

Holomorphic invariant conditions:

$$H_0^T J H_0 = z^k J + O(z^{k-1}) .$$

(33)

$G' = SO(2N + 1)$:

$$H_0^T J H_0 = z^{2k} J + O(z^{2k-1}) .$$

(34)
Special points

\[ H_0(z) = z^{\nu 1_N + \nu \mathcal{H}_\alpha} \in U(1)^C \times G'^C, \]

\( \mathcal{H}_\alpha \): Cartan generators of \( g' \), \( \alpha = 1, \ldots, \text{rank}(G') \)

\[ \nu = \frac{k}{n_0}. \]

Single valued condition:

\[ (\nu 1_N + \nu \mathcal{H}_\alpha)_{ii} \in \mathbb{Z}_{\geq 0} \quad \forall i, \quad \Rightarrow \nu + \nu \alpha \mu_{\alpha}^{(j)} \in \mathbb{Z}_{\geq 0} \quad \forall i, \]

\( \mu_{\alpha}^{(j)} \): weight vector of \( G' \).
Quantization with respect to the dual group

Quantization condition:

\[ \vec{\nu} \cdot \vec{\alpha}^{(j)} \in \mathbb{Z}, \]

\( \alpha^{(j)} \): root vectors of \( G' \).

Formally Goddard-Nuyts-Olive-Weinberg quantization condition.

Solution:

\[ \vec{\nu} = 2\vec{\mu}, \quad (36) \]

\( \vec{\mu} \): any weight vector of dual group, dual group’s root vectors

\[ \vec{\alpha}^* = \frac{\vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}}. \quad (37) \]

No conceptual problems in quantizing zero-modes due to exact color+flavor symmetry.
\[ \mathbb{Z}_2 \text{ parity for } G' = SO(N) \]

First homotopy group
\[ \pi_1 \left( \frac{U(1) \times SO(N)}{Z_{n_0}} \right) = \mathbb{Z} \times \mathbb{Z}_2 , \]

\[ \begin{aligned} n_0 &= 1 , \quad N \text{ odd} \\ n_0 &= 2 , \quad N \text{ even} \end{aligned} \]  

(38)

Moduli space is disconnected:
Special points for $k = 1$ vortex with $G' = SO, USp$. 

$SO(2)$

$USp(2)$

$SO(3)$

$USp(4)$

$SO(5)$
Dark grey points have positive $\mathbb{Z}_2$-charge, white ones have negative.
Moduli space of vortices

The moduli space of vortices: completely described by moduli matrix

\[ \mathcal{M} = \{H_0(z)\} \mathbin{/} U(N)^\mathbb{C}. \] (39)

Some examples: \( k = 1 \)

\[ \mathcal{M}_{USp(2M)} = \mathbb{C} \times \frac{USp(2M)}{U(M)}, \] (40)

\[ \mathcal{M}_{SO(2M)} = \left( \mathbb{C} \times \frac{SO(2M)}{U(M)} \right)_+ \cup \left( \mathbb{C} \times \frac{SO(2M)}{U(M)} \right)_-, \] (41)
Connectedness of the moduli space

\[SO(2) \quad \begin{array}{c}
-\frac{1}{2} \\
\frac{1}{2}
\end{array} \quad USp(2) \quad \begin{array}{c}
\frac{1}{2} \\
-\frac{1}{2}
\end{array} \]

\[SU(2)/U(1)_- \quad \begin{array}{c}
-\frac{1}{2}, -\frac{1}{2}
\end{array} \quad \begin{array}{c}
\frac{1}{2}, \frac{1}{2}
\end{array} \quad SO(4) \quad \begin{array}{c}
\frac{1}{2}, -\frac{1}{2}
\end{array} \quad USp(4) \quad \begin{array}{c}
\frac{1}{2}, \frac{1}{2}
\end{array} \]

\[SU(2)/U(1)_+ \quad \begin{array}{c}
\frac{1}{2}, \frac{1}{2}
\end{array} \quad \begin{array}{c}
\frac{1}{2}, -\frac{1}{2}
\end{array} \]

\[CP^1 \quad \begin{array}{c}
-\frac{1}{2}, -\frac{1}{2}
\end{array} \quad \begin{array}{c}
\frac{1}{2}, -\frac{1}{2}
\end{array} \]
The $k = 1$ odd $SO$ “$\Leftrightarrow$” $k = 2$ even:
Differences

\[ \begin{align*}
(1, 1) & \quad (1, -1) \\
(-1, -1) & \quad (-1, 1) \\
(0, 1) & \quad (0, -1) \\
(1, 0) & \quad (-1, 0) \\
(0, 0) & \quad (0, 0)
\end{align*} \]

\[ \begin{align*}
(\frac{1}{2}, \frac{1}{2}) & \quad (\frac{1}{2}, -\frac{1}{2}) \\
(-\frac{1}{2}, \frac{1}{2}) & \quad (-\frac{1}{2}, -\frac{1}{2})
\end{align*} \]

\[ \begin{align*}
SO(4) & \quad SO(5)
\end{align*} \]

\[ \begin{align*}
k = 2, \; SO(2) & \quad k = 2, \; USp(2) \\
k = 1, \; SO(3)
\end{align*} \]

\[ \begin{align*}
\mathbb{CP}^1 & \quad \mathbb{CP}^1 \\
\simeq (\mathbb{CP}^1)^2 / S_2 & \quad \simeq\mathbb{CP}^1
\end{align*} \]
$SO(4)$ higher windings

$\begin{align*}
  k &= 1 \\
  k &= 2 \\
  k &= 3 \\
  k &= 4 \\
  k &= 5
\end{align*}$
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Existence and uniqueness

1. Result of our index calculation with generic gauge group

\[ \# \text{ bosonic zero-modes} \leftrightarrow \# \text{ moduli in } H_0(z) \]

2. Strong gauge coupling limit \( \Rightarrow \) non-linear \( \sigma \)-model analytic solution is obtained
Strong gauge coupling limit

\(SU(N)\):

\[
0 = \frac{1}{\omega} \text{Tr} \Omega_0 \Omega'^{-1} - v^2 ,
\]

\[
0 = \Omega_0 \Omega'^{-1} - \frac{1_N}{N} \text{Tr} \Omega_0 \Omega'^{-1} ,
\]

\(SO, USp\):

\[
0 = \Omega_0 \Omega'^{-1} - J^\dagger \left( \Omega_0 \Omega'^{-1} \right)^T J ,
\]

Solution for \(SU(N)\):

\[
\omega_\infty = \frac{N}{v^2} \left( \text{det} \Omega_0 \right)^{\frac{1}{N}} , \quad \Omega'_\infty = \left( \text{det} \Omega_0 \right)^{-\frac{1}{N}} \Omega_0 ,
\]

Solution for \(SO, USp\):

\[
\omega_\infty = \frac{1}{v^2} \text{Tr} \sqrt{M^\dagger M} , \quad \Omega'_\infty = \frac{1_N}{\sqrt{M^\dagger M}} H_0^T(z) H_0^\dagger(z) .
\]

(42, 43, 44, 45, 46)
The NL$\sigma$M is integrable.

Except for the local vortex which is mapped to a point
⇒ small lump singularity.
Kähler quotient

\[ U(N): \]
\[ \mathcal{L} = \text{Tr} \int d^4 \theta \ \left\{ QQ^\dagger e^{-V} + \xi V \right\}, \quad (47) \]
\[ \sim \int d^4 \theta \ \xi \log \det QQ^\dagger, \quad (48) \]

\[ SO, USp: \]
\[ \mathcal{L} = \text{Tr} \int d^4 \theta \ \left\{ QQ^\dagger e^{-V'} e^{-V_e} + \xi V_e \right\}, \quad (49) \]

\[ e^{-V'} \in SO, USp. \]

Difficult calculation.
Kähler quotient for $SO, USp$

Relax the algebra $e^{-V'} \in SL(N, \mathbb{C})$, introduce Lagrange multipliers $\lambda$.

$$
\mathcal{L} = \text{Tr} \int d^4 \theta \left\{ QQ^\dagger e^{-V'} e^{-V_e} + \lambda \left( e^{-V'^T} J e^{-V'} - J \right) + \xi V_e \right\},
$$

$$
\sim \int d^4 \theta \xi \log \text{Tr} \sqrt{MM^\dagger}, \quad (50)
$$

$M = Q^T J Q$ is the meson field.
A similar construction has been made for the hyper-Kähler case ($\mathcal{N} = 2$).
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Fractional vortices of the first type

\[ M \sim S^2 \text{ with two singularities} \]

\[ \mathbb{Z}_m \text{ singularity} \]

\[ \mathbb{Z}_n \text{ singularity} \]

Now to avoid a singular field configuration the 2-cycle should wrap \( m \times n \) times

\[ \text{winding m-lump} \]

\[ \text{winding n-lump} \]

\[ \text{vev} \]

\[ \text{vev} \]
Formally we can write

\[ R(z) = R_{\text{rev}} \frac{\prod_{j=1}^{n} (z - z_j)^{u_j}}{\prod_{i=1}^{n} (z - z_i)^{u_i}} \]
Fractional vortices of the second type
The droplet model

\[
\begin{array}{c|cc}
& U(1) \\
\hline
A & 2 \\
B & 1 \\
\end{array}
\]

\[H = \begin{pmatrix} A \\ B \end{pmatrix}. \tag{51}\]

\[
\phi(z) = \phi_{\text{vev}} \frac{(z - z_{1}^{S})(z - z_{2}^{S})}{(z - z_{1}^{N})^2}. \tag{52}\]
Example: $SO(6)$ theory

\[
\begin{array}{c}
1 \\
\frac{1}{3} + \frac{1}{3} + \frac{1}{3}
\end{array}
\]
Plan of the talk

Introductory part

- motivation
- crash course on vortices
- summary of recent results in the field

Results

- $SO, USp$ theories
- $NL\sigma M$ lumps
- fractional vortices
  - Chern-Simons vortices
Yang-Mills-Chern-Simons-Higgs theory

$\mathcal{N} = 2$ (4 supercharges) in $d = 2 + 1$ dimensions with the gauge group $G = U(1) \times G'$, where $G'$ is a simple group.

$$\mathcal{L}_{\text{YMCSH}} = - \frac{1}{4g^2} (F_{\mu\nu}^a)^2 - \frac{1}{4e^2} (F_{\mu\nu}^0)^2 - \frac{\kappa}{8\pi} \epsilon^{\mu\nu\rho} A^0_\mu \partial_\nu A^0_\rho$$

$$- \frac{\mu}{8\pi} \epsilon^{\mu\nu\rho} \left( A^a_\mu \partial_\nu A^a_\rho - \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho \right)$$

$$+ \frac{1}{2g^2} (D_\mu \phi^a)^2 + \frac{1}{2e^2} (\partial_\mu \phi^0)^2 + \text{Tr} \left( D_\mu H \right) \left( D^\mu H \right)^\dagger$$

$$- \text{Tr} \left| \phi H - Hm \right|^2 - \frac{g^2}{2} \left( \text{Tr} \left( HH^\dagger t^a \right) - \frac{\mu}{4\pi} \phi^a \right)^2$$

$$- \frac{e^2}{2} \left( \text{Tr} \left( HH^\dagger t^0 \right) - \frac{\kappa}{4\pi} \phi^0 - \frac{1}{\sqrt{2N}} \xi \right)^2$$,
Integrate out the adjoint fields

Strong gauge coupling limit $e, g \to \infty$:

\[ \phi^a = \frac{4\pi}{\mu} \text{Tr} \left( HH^\dagger t^a \right) , \]

\[ \phi^0 = \frac{4\pi}{\kappa} \frac{1}{\sqrt{2N}} \left[ \text{Tr} \left( HH^\dagger \right) - \xi \right] . \]
Chern-Simons-Higgs theory

\[ \mathcal{L}_{\text{CSH}} = -\frac{\mu}{8\pi} \epsilon^{\mu\nu\rho} \left( A_{\mu}^{a} \partial_{\nu} A_{\rho}^{a} - \frac{1}{3} f^{abc} A_{\mu}^{a} A_{\nu}^{b} A_{\rho}^{c} \right) - \frac{\kappa}{8\pi} \epsilon^{\mu\nu\rho} \left( A_{\mu}^{0} \partial_{\nu} A_{\rho}^{0} \right) + \text{Tr} \left( \mathcal{D}_{\mu} H \right)^{\dagger} \left( \mathcal{D}^{\mu} H \right) \]

\[ - 4\pi^2 \text{Tr} \left\{ \frac{1}{N} \left( \text{Tr} \left( H H^{\dagger} \right) - \xi \right) + \frac{2}{\mu} \text{Tr} \left( H H^{\dagger} t^a \right) t^a \right\} H \right|^{2}, \]

(55)
Master equations

\[
\bar{\partial} \left[ \Omega' \partial \Omega'^{-1} \right] = \frac{2\pi^2}{N \kappa \mu} \left( \text{Tr} \left( \Omega_0 \Omega^{-1} \right) - \xi \right) \langle \Omega_0 \Omega^{-1} \rangle_J \\
+ \frac{\pi^2}{\mu^2} \left\langle \left( \Omega_0 \Omega^{-1} \right)^2 \right\rangle_J ,
\]

(56)

\[
\bar{\partial} \partial \log \omega = -\frac{4\pi^2}{N^2 \kappa^2} \text{Tr} \left( \Omega_0 \Omega^{-1} \right) \left( \text{Tr} \left( \Omega_0 \Omega^{-1} \right) - \xi \right) \\
- \frac{2\pi^2}{N \kappa \mu} \text{Tr} \left( \Omega_0 \Omega^{-1} \langle \Omega_0 \Omega^{-1} \rangle_J \right) .
\]

(57)

\[
\langle X \rangle_J \equiv X - J^\dagger X^T J .
\]
\[ E_r^0 \]

\[ E_r^{\text{NA}} \]

\[ -\kappa F_{12} \]

\[ -\mu F_{12} \]

\[ -\kappa F_{12}^{\text{NA}} \]

\[ -\mu F_{12}^{\text{NA}} \]
Abelian magnetic field

Non-Abelian magnetic field

Abelian magnetic field

Non-Abelian magnetic field
Opposite sign of couplings

\[ r e^{-(\psi+\chi)/2} \]

\[ e^{-(\psi-\chi)/2} \]

\[ \kappa = -4, \mu = 2 \]
\[ \kappa = -2, \mu = 2 \]
\[ \kappa = -1, \mu = 2 \]
\[ \kappa = -4, \mu = 2 \]
\[ \kappa = -2, \mu = 2 \]
\[ \kappa = -1, \mu = 2 \]
Fractional Chern-Simons vortex

\[ SO(2M): \]

\[
H_0 = \begin{pmatrix}
\begin{array}{ccc|ccc}
z - z_1 & \ddots & & \ddots & c_1 & \\
0 & & \ddots & & & \\
\vdots & \ddots & & & \ddots & \\
0 & \ddots & & & 1 & \\
& & & & & c_M
\end{array}
\end{pmatrix},
\]

(58)

\[
\Omega' = \text{diag} \left( e^{\chi_1}, \ldots, e^{\chi_M}, e^{-\chi_1}, \ldots, e^{-\chi_M} \right),
\]

(59)

\[
\omega = e^{\psi}.
\]

(60)
\[ \delta \chi_m = \frac{2|c_m|^2}{m^2_{\mu}} |z|^{-4} \quad \text{Asymptotic profiles} \]

\[ + \frac{2|c_m|^2}{m^2_{\mu}} \left[ 3 \left( \frac{z_m}{z} + \frac{\bar{z}_m}{\bar{z}} \right) - \frac{1}{M} \sum_{n=1}^{M} \left( \frac{z_n}{z} + \frac{\bar{z}_n}{\bar{z}} \right) \right] |z|^{-4} + \mathcal{O} \left( |z|^{-6} \right) \]

\[ \delta \psi = \frac{1}{M m^2_\kappa} \left( \sum_{n=1}^{M} \left( |z_n|^2 + 2|c_n|^2 \right) - \frac{1}{M} \left| \sum_{n=1}^{M} z_n \right|^2 \right) |z|^{-4} \]

\[ + \frac{1}{M m^2_\kappa} \left[ \frac{1}{2} \sum_{n=1}^{M} \left( |z_n|^2 + 4|c_n|^2 \right) \left( \frac{z_n}{z} + \frac{\bar{z}_n}{\bar{z}} \right) \right. \]

\[ - \frac{1}{2M} \left( \sum_{n=1}^{M} \frac{z_n^2}{z} \sum_{n'=1}^{M} \bar{z}_{n'} + \sum_{n=1}^{M} z_n \sum_{n'=1}^{M} \frac{\bar{z}_{n'}^2}{\bar{z}} \right) \]

\[ + \frac{1}{M} \left( \sum_{n=1}^{M} \left( |z_n|^2 + 2|c_n|^2 \right) - \frac{1}{M} \left| \sum_{n=1}^{M} z_n \right|^2 \right) \sum_{n'=1}^{M} \left( \frac{z_{n'}}{z} + \frac{\bar{z}_{n'}}{\bar{z}} \right) \]

\[ + \mathcal{O} \left( |z|^{-6} \right), \]
Effective size

$$|c_{\text{effective}}|^2 = \frac{1}{2M} \sum_{n=1}^{M} \left( |z_n|^2 + 2|c_n|^2 \right) - \frac{1}{2M^2} \left| \sum_{n=1}^{M} z_n \right|^2,$$
Figure – fractional Chern-Simons vortex
Future developments

• monopole-vortex systems
• the GNOW-duality
• group theory of vortices (in preparation)
• Yang-Mills-Chern-Simons-Higgs (in preparation)
• non-BPS corrections / stability
• $Q$-lumps in $SO, U Sp$ theories
• quantum corrections to the $\mathcal{N} = 1$ Kähler quotients
• Ricci-flat Calabi-Yau metrics
• D-brane constructions
• knotted solitons
• quantized vortices
• domain wall systems in $SO, U Sp$ theories
• the mass deformed theories
• etc.
Thanks for your attention