Variational Calculation of $^4\text{He}$ Tetramer Ground and Excited States Using a Realistic $^4\text{He}–^4\text{He}$ Potential

E. Hiyama
RIKEN, Nishina Center

Collaborator: M. Kamimura (Kyushu/RIKEN)
As you might notice, one of the important keywords of this Session II–d is "Universality in 4-body systems".

I discuss it from the viewpoint of universality (similarity) lying between atomic 4-body systems and nuclear 4-body systems with large scattering length.
Very recently, we published two papers on the $^4\text{He}$ tetramer:

1) "Variational calculation of $^4\text{He}$ tetramer ground and excited states using a realistic $^4\text{He}–^4\text{He}$ potential", E.H. and M. Kamimura, Phys. Rev. A 85 (2012) 022502

2) "Linear correlations between $^4\text{He}$ trimer and tetramer energies calculated with various realistic $^4\text{He}–^4\text{He}$ potentials"  

The details will be presented in Poster PS–64.
Section 1
Introduction
Similarity in energy levels of atomic and nuclear 4-body systems with large scattering length
(the two lowest-lying states)

Atom

\[ ^4\text{He} \quad ^4\text{He} \quad ^4\text{He} \quad ^4\text{He} \]

Nucleus

\[ ^4\text{He} \quad ^4\text{He} \quad ^4\text{He} \quad ^4\text{He} \]

\[ N \quad N \quad N \quad N \]

\[ \alpha \quad \alpha \quad \alpha \quad \alpha \]

4He - atom tetramer

^4He nucleus
(4 - Nucleon)

^16O nucleus
(4 - \alpha )

\[ \alpha = ^4\text{He} \text{ nucleus} \]
Similarity in energy levels of atomic and nuclear 4-body systems with large scattering length
(the two lowest-lying states)

<table>
<thead>
<tr>
<th>Atom</th>
<th>Nucleus</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^4\text{He} + \text{trimer}$ 0.0</td>
<td>$\alpha + ^{12}\text{C}$ 0.0</td>
</tr>
<tr>
<td>0$^+$</td>
<td>0$^+$</td>
</tr>
<tr>
<td>- 1.0 mK</td>
<td>- 1.11 MeV</td>
</tr>
<tr>
<td>$^4\text{He}$-atom tetramer</td>
<td>$^4\text{He}$ nucleus</td>
</tr>
<tr>
<td></td>
<td>(4-Nucleon)</td>
</tr>
<tr>
<td></td>
<td>$^{16}\text{O}$ nucleus</td>
</tr>
<tr>
<td></td>
<td>(4- $\alpha$)</td>
</tr>
<tr>
<td>- 432.6 mK</td>
<td>- 20.58 MeV</td>
</tr>
<tr>
<td></td>
<td>- 7.16 MeV</td>
</tr>
</tbody>
</table>
Do we have

**Similarity in wave functions of atomic and nuclear 4-body systems with large scattering length?**

Overlap function

\[ O_4^{(v)}(z) = \langle \Psi_3^{(0)} | \Psi_4^{(v)} \rangle_{x,y} \]

\[ \langle \Psi_3^{(0)} | \Psi_4^{(v)} \rangle_{x,y} = \langle \Psi_3^{(0)} | \Psi_4^{(v)} \rangle \]

(E. Hiyama et al., PRC 70 (2004))

**Overlap function for \(^4\)He nucleus**

How is in \(^4\)He-atom tetramer?
Correlation between the 3N ($^3$H) and 4N ($^4$He) binding energies for different NN potentials is approximately linear. This line is called Tjon line (well known in nuclear few-body physics).

Similarity in Tjon line of atomic and nuclear 4-body systems with large scattering length?

How is this type of correlation in $^4$He-atom tetramer?

The slope is universal?
6 types of the correlations in \(^4\text{He}-\text{atom tetramer}\) (generalized Tjon lines)

Atomic generalized Tjon lines (6 types)

\[^4\text{He} \text{ trimer}\] \[^4\text{He} \text{ tetramer}\]
In this talk, we will discuss about these similarity.

Similarity in wave functions of atomic and nuclear 4-body systems with large scattering length?

Similarity in Tjon line of atomic and nuclear 4-body systems with large scattering length?
Our few-body calculation method

**Gaussian Expansion Method (GEM), since 1987**

- A variational method using Gaussian basis functions
- Take all the sets of Jacobi coordinates

**Developed by Kyushu Univ. Group, Kamimura and his collaborators.**

Review article:
E. Hiyama, M. Kamimura and Y. Kino,
Prog. Part. Nucl. Phys. 51 (2003), 223.

**High-precision calculations** of various 3- and 4-body systems:

- Exotic atoms / molecules, Light hypernuclei,
- 4-nucleon systems, 3-quark systems,
- multi-cluster structure of light nuclei,
We employ various $^4$He–$^4$He realistic potentials:


ii) HFD-B3-I1 (1997), SAPT97 (1997), CCSAPT07 (2007),

iii) PCKLJS (2010)

ii) The second group include the retardation correction.

iii) the PCKLJS is the currently most accurate potential that includes corrections for retardation, non-adiavatic, relativistic and QED effects.

The conversion constant $\frac{\hbar^2}{m} = 12.11928$ K Å$^2$ is taken.
# Binding energies for 7 realistic potentials

<table>
<thead>
<tr>
<th>Potential</th>
<th>Trimer</th>
<th>Tetramer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B_3^{(0)}$ (mK)</td>
<td>$B_4^{(0)}$ (mK)</td>
</tr>
<tr>
<td>LM2M2</td>
<td>126.50</td>
<td>559.22</td>
</tr>
<tr>
<td>TTY</td>
<td>126.45</td>
<td>558.70</td>
</tr>
<tr>
<td>HFD-B3-FCI1</td>
<td>129.00</td>
<td>566.12</td>
</tr>
<tr>
<td>CCSAPT07</td>
<td>131.01</td>
<td>571.67</td>
</tr>
<tr>
<td><strong>PCKLJS</strong></td>
<td>131.84</td>
<td>573.90</td>
</tr>
<tr>
<td>HFD-B</td>
<td>133.08</td>
<td>577.34</td>
</tr>
<tr>
<td>SAPT96</td>
<td>134.02</td>
<td>580.01</td>
</tr>
</tbody>
</table>

These values are used for plotting the generalized Tjon lines.
The currently most accurate PCKLJS potential (2010)

M. Przybytek et al., Phys. Rev. Lett. 104 (2012) 183003, It includes the adiabatic, relativistic, QED and residual retardation corrections:

\[ V(r) = V_{BO}(r) + V_{ad}(r) + V_{rel}(r) + V_{QED}(r) + \text{r.c.} \]

These values by PCKLJS (a) ... (h) are used for plotting the generalized Tjon lines.

<table>
<thead>
<tr>
<th>PCKLJS potential</th>
<th>Dimer</th>
<th>Trimer</th>
<th>Tetramer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B_2$ (mK)</td>
<td>$\langle r \rangle$ (Å)</td>
<td>$B_3^{(0)}$ (mK)</td>
</tr>
<tr>
<td>(a) $V_{BO}$</td>
<td>1.7181</td>
<td>45.77</td>
<td>133.43</td>
</tr>
<tr>
<td>(b) $V_{BO} + \text{r.c.}$</td>
<td>1.5549</td>
<td>47.92</td>
<td>130.85</td>
</tr>
<tr>
<td>(c) $V_{BO} + V_{ad}$</td>
<td>1.8160</td>
<td>44.62</td>
<td>134.96</td>
</tr>
<tr>
<td>(d) $V_{BO} + V_{ad} + \text{r.c.}$</td>
<td>1.6482</td>
<td>46.65</td>
<td>132.37</td>
</tr>
<tr>
<td>(e) $V_{BO} + V_{ad} + V_{rel}$</td>
<td>1.5896</td>
<td>47.43</td>
<td>131.44</td>
</tr>
<tr>
<td>(f) $V_{BO} + V_{ad} + V_{rel} + \text{r.c.}$</td>
<td>1.6105</td>
<td>47.15</td>
<td>131.76</td>
</tr>
<tr>
<td>(g) $V_{BO} + V_{ad} + V_{rel} + V_{QED}$</td>
<td>1.6200</td>
<td>47.02</td>
<td>131.90</td>
</tr>
<tr>
<td>(h) $V_{BO} + V_{ad} + V_{rel} + V_{QED} + \text{r.c.}$</td>
<td>1.6154</td>
<td>47.09</td>
<td>131.84</td>
</tr>
</tbody>
</table>
Universality in overlap function

\[ O_{4}^{(v)}(z) = \langle \Psi_{3}^{(0)} | \Psi_{4}^{(v)} \rangle_{x,y} = \sum_{n=0}^{4} \langle \Psi_{3}^{(0)} | \Psi_{n}^{(v)} \rangle_{x,y} \]

Size ratio = 1 : 400 000

E. Hiyama et al., PRC 70 (2004)

E. Hiyama & M.K., PRA 85 (2012)
Universality in Tjon line
(Linear correlation between $B_3$ and $B_4$)

Atomic generalized Tjon lines (6 types)

$B_3^{(1)} \quad B_4^{(1)}$

$B_3^{(0)} \quad B_4^{(0)}$

$0^+ \quad 0^+$

$4^\text{He}$ trimer $4^\text{He}$ tetramer

Correlations

$B_3^{(1)} - B_4^{(1)}$, $B_3^{(0)} - B_4^{(0)}$, $B_3^{(0)} - B_4^{(1)}$, $B_3^{(1)} - B_4^{(0)}$, $B_4^{(0)} - B_4^{(1)}$

i) 7 results (●) calculated with HFD–B, LM2M2, TTY, HFD–B3–FCI1, SAPT97, CCSAPT07, PCKLJS.

ii) 7 results (□) calculated with PCKLJS (a) to (g).
2-body realistic forces

A. Nogga et al., PRL 85 (2000)

E. Hiyama & M. K, PRA 85 (2012)

Slope = 4.8

Slope = 4.778
Redlines are obtained by the linear least square fit.

\[ B_3^{(1)} - B_4^{(1)} \]

\[ B_3^{(0)} - B_4^{(1)} \]

\[ B_3^{(1)} - B_4^{(0)} \]

\[ B_3^{(0)} - B_4^{(0)} \]

\[ B_3^{(0)} - B_3^{(1)} \]

\[ B_4^{(0)} - B_4^{(1)} \]
In the left-side 3 figures, the solid line shows the universal scaling curve obtained by the leading-order effective theory for $^4$He atoms 


The 3 correlations are well understood by the universal scaling curve (deviation is only 2–3%).
1) the ground and excited states of $^4$He trimer, and
2) the excited state of $^4$He tetramer, (which appear in the figures) are Efimov states.

if the binding energies move along their universal scaling curves (solid line)
under any small deformation of the two-body potential, the states are Efimov state.
On the other hand, in the right-side 3 figures, deviation of the universal scaling curve is significant.

As you might notice, in all the figures, there appears $B_4^{(0)}$, the binding energy of the tetramer ground state.

Then, the deviation, roughly 15%, is due to the fact that the leading-order effective theory underestimates $B_4^{(0)}$ by about 15%.

Therefore, I think, the ground state of $^4$He tetramer might not be an Efimov state.
Concluding remark

How about 5-body $^4\text{He}$ system?

$^4\text{He}+^4\text{He}+^4\text{He}+^4\text{He}+^4\text{He}$

Tetramer+$^4\text{He}$

The excited state might be Efimov state.

Deviation of universal scaling curve might be more significant.
Thank you!