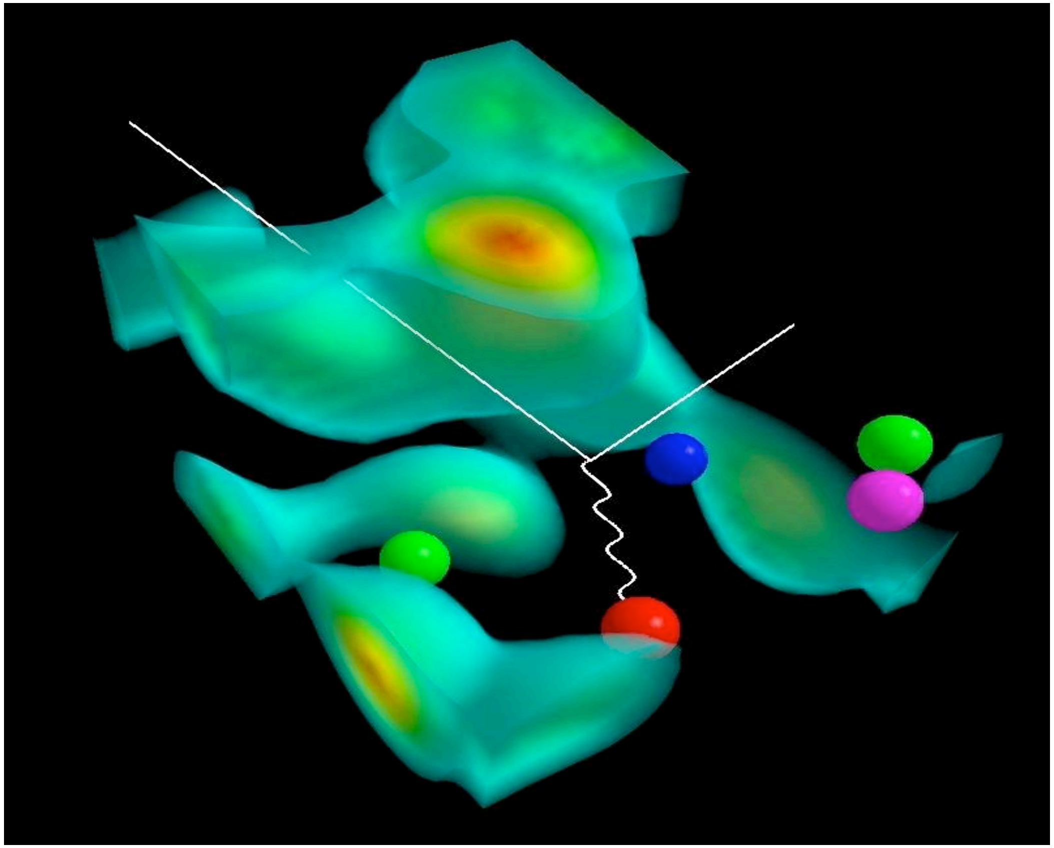
Distribution of Spin and Orbital Angular Momentum in the Proton



Anthony W. Thomas

6^{th} Int. Conference on Perspectives in Hadronic Physics
ICTP Trieste: May 12^{th} 2008

Outline

- A reminder: the proton spin crisis

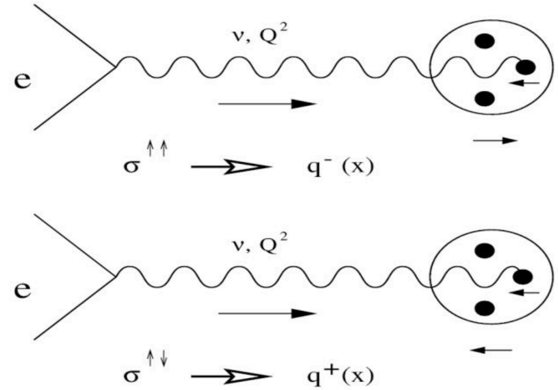
- Progress over the last 20 years

- The resolution of the problem
  - one-gluon-exchange
  - the pion cloud
  - input from lattice QCD

- Lattice QCD

- GPDs at JLab
  – at 12 GeV
  – recent results

The EMC "Spin Crisis"



$$g_1(x) = \frac{1}{2}\sum_q e_q^2\left(q^+(x) - q^-(x)\right) = \frac{1}{2}\sum_q e_q^2 \Delta q(x)$$

Up to standard pQCD coefficients (series in $\alpha_s(Q^2)$):

$$\int_0^1 dx\, g_1^p(x) = (\Delta u - \Delta d)/12 + (\Delta u + \Delta d - 2\Delta s)/36$$

$$+ (\Delta u + \Delta d + \Delta s)/9$$

(up to QCD radiative corrections)

g^3_A : from β decay of n

g^8_A : hyperon β decay

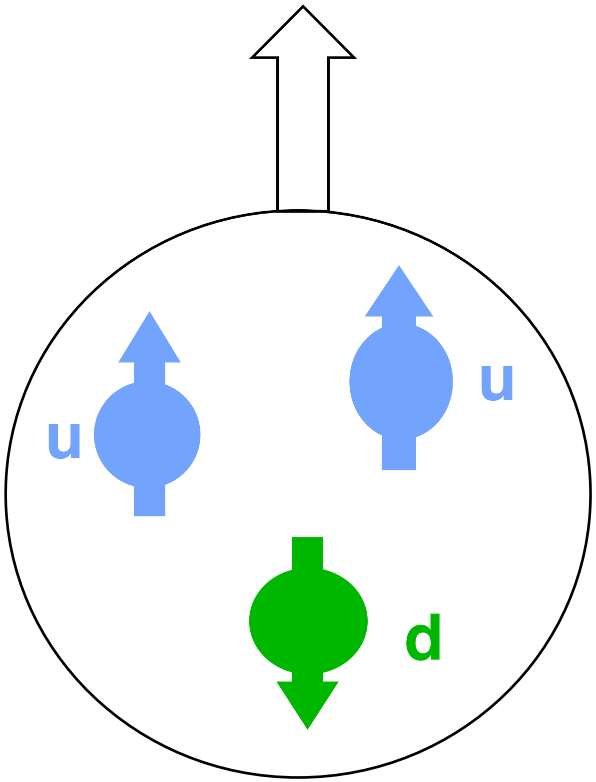
naively fraction of proton **'spin' carried by its quarks**

$$\Sigma_{inv} \equiv \Sigma(Q^2 = \infty)$$

Δ u ≡ fraction of proton spin carried by u and anti-u quarks, etc..

What do we expect ?

Most quark models start with 3 quarks in the 1s-state of a confining potential: proton spin is ALL carried by its quarks $\Rightarrow \Sigma = 100\%$



N.B. Given low values of $m_{u,d}$ the quark motion is relativistic and lower Dirac components have spin down $\Rightarrow \Sigma \sim 65\%$

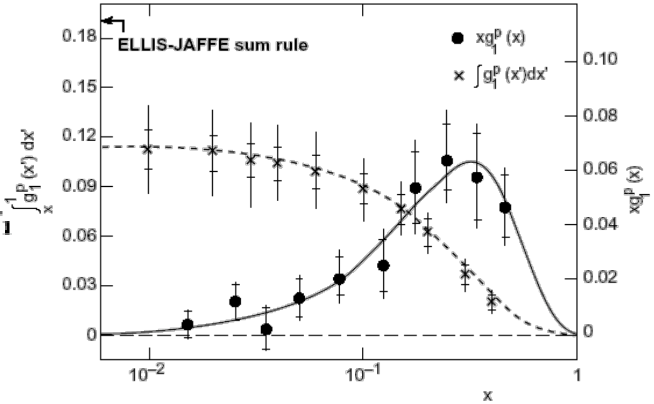
Ancient History of the Spin Crisis

- **EMC Spin Paper: 22 Dec 87 - 19 May 88**
- Brodsky et al. Skyrme: 22 Feb 88 - 19 May 88
- Schreiber-Thomas CBM: 17 May 88 - 8 Dec 88
- Myhrer-Thomas OGE: 13 June 88 - 1 Sept 88

(neither paper could explain reduction to only 14%!)

- Efremov-Teryaev Anomaly: 25 May 88
- Altarelli-Ross Anomaly: 29 June 88 - 29 Sept 88

A MEASUREMENT OF THE SPIN ASYMMETRY AND DETERMINATION OF THE STRUCTURE FUNCTION g_1 IN DEEP INELASTIC MUON–PROTON SCATTERING



European Muon Collaboration

Aachen, CERN, Freiburg, Heidelberg, Lancaster, LAPP (Annecy), Liverpool, Marseille, Mons, Oxford, Rutherford, Sheffield, Turin, Uppsala, Warsaw, Wuppertal, Yale

J. ASHMAN [a], B. BADELEK [b,1], G. BAUM [c,2], J. BEAUFAYS [d], C.P. BEE [e], C BENCHOUK [f],

(93 authors)

The spin asymmetry in deep inelastic scattering of longitudinally polarised muons by longitudinally polarised protons has been measured over a large x range $(0.01 < x < 0.7)$. The spin-dependent structure function $g_1(x)$ for the proton has been determined and its integral over x found to be $0.114 \pm 0.012 \pm 0.026$, in disagreement with the Ellis–Jaffe sum rule. Assuming the validity of the Bjorken sum rule, this result implies a significant negative value for the integral of g_1 for the neutron. These values for the integrals of g_1 lead to the conclusion that the total quark spin constitutes a rather small fraction of the spin of the nucleon.

$\Sigma = 14 \pm 3 \pm 10$ % :

i.e. 86% of spin of p NOT carried by its quarks

Ancient History of the Spin Crisis

- EMC Spin Paper: 22 Dec 87 - 19 May 88
- Brodsky et al. Skyrme: 22 Feb 88 - 19 May 88
- Schreiber-Thomas CBM: 17 May 88 - 8 Dec 88
- Myhrer-Thomas OGE: 13 June 88 - 1 Sept 88

(neither paper could explain reduction to only 14%!)

- Efremov-Teryaev Anomaly: 25 May 88
- Altarelli-Ross Anomaly: 29 June 88 - 29 Sept 88

Ancient History of the Spin Crisis

- EMC Spin Paper: 22 Dec 87 - 19 May 88
- Brodsky et al. Skyrme: 22 Feb 88 - 19 May 88
- Schreiber-Thomas CBM: 17 May 88 - 8 Dec 88
- Myhrer-Thomas OGE: 13 June 88 - 1 Sept 88

(neither paper could explain reduction to only 14%!)

- **Efremov-Teryaev Anomaly: 25 May 88**
- **Altarelli-Ross Anomaly: 29 June 88 - 29 Sept 88**

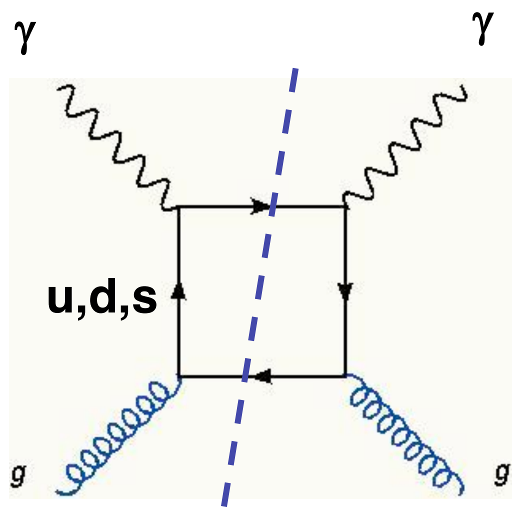
ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2-88-287

A.V.Efremov, O.V.Teryaev*

SPIN STRUCTURE OF THE NUCLEON AND TRIANGLE ANOMALY

Submitted to "Nuclear Physics"



25 May 1988

Volume 212, number 3 PHYSICS LETTERS B 29 September 1988

THE ANOMALOUS GLUON CONTRIBUTION TO POLARIZED LEPTOPRODUCTION

G. ALTARELLI and G.G. ROSS [1]
CERN, CH-1211 Geneva 23, Switzerland

Received 29 June 1988

We show that, due to the anomaly, the gluon contribution to the first moment of the polarized proton structure function, as measured in deep inelastic scattering, is not suppressed by a power of the strong coupling evaluated at a large scale. As a result, the EMC result for the first moment of polarized proton electroproduction is consistent with a large quark spin component.

$$\Sigma_{\text{naïve}} \rightarrow \Sigma_{\text{naïve}} - N_f \frac{\alpha_s (Q^2)}{2\pi} \Delta G (Q^2)$$

and

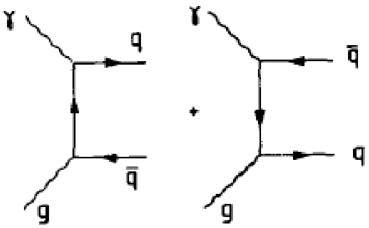


Fig. 1. Diagrams contributing to a finite mixing of order α_s between g_1^p and the polarized gluon parton density.

QCD evolution ⇒ $\alpha_s(Q^2)\, \Delta G(Q^2)$ does not vanish as $Q^2 \rightarrow \infty$

and polarized gluons would resolve crisis **HOW MUCH?**

Scale of the Gluon Contribution

At 3 GeV^2 $\alpha_s \sim 0.3$

and $N_f = 3$, so IF all of the

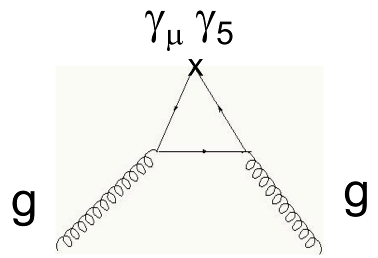
N spin carried by quarks is

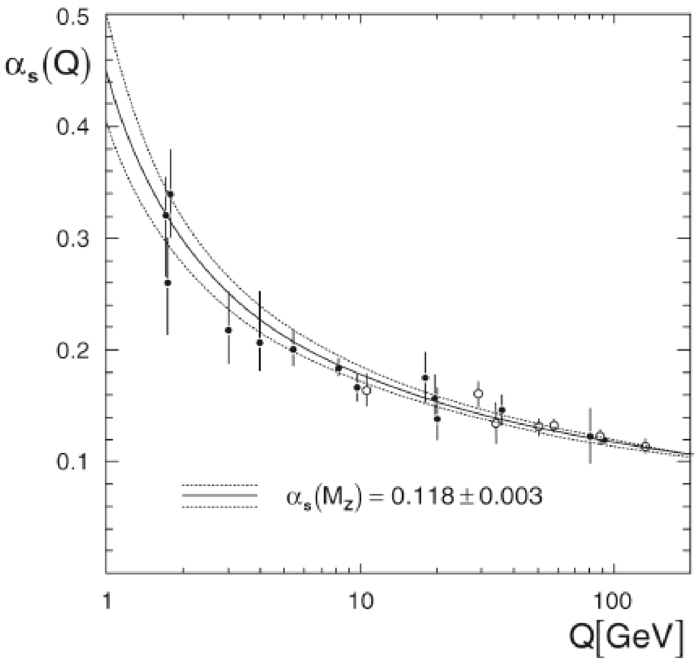
cancelled by gluons:

$$\Delta G = + \frac{2 * \pi * 1}{3 * 0.3} \sim +6$$

…actually $\Delta G \sim + 4$ better

- a truly remarkable result



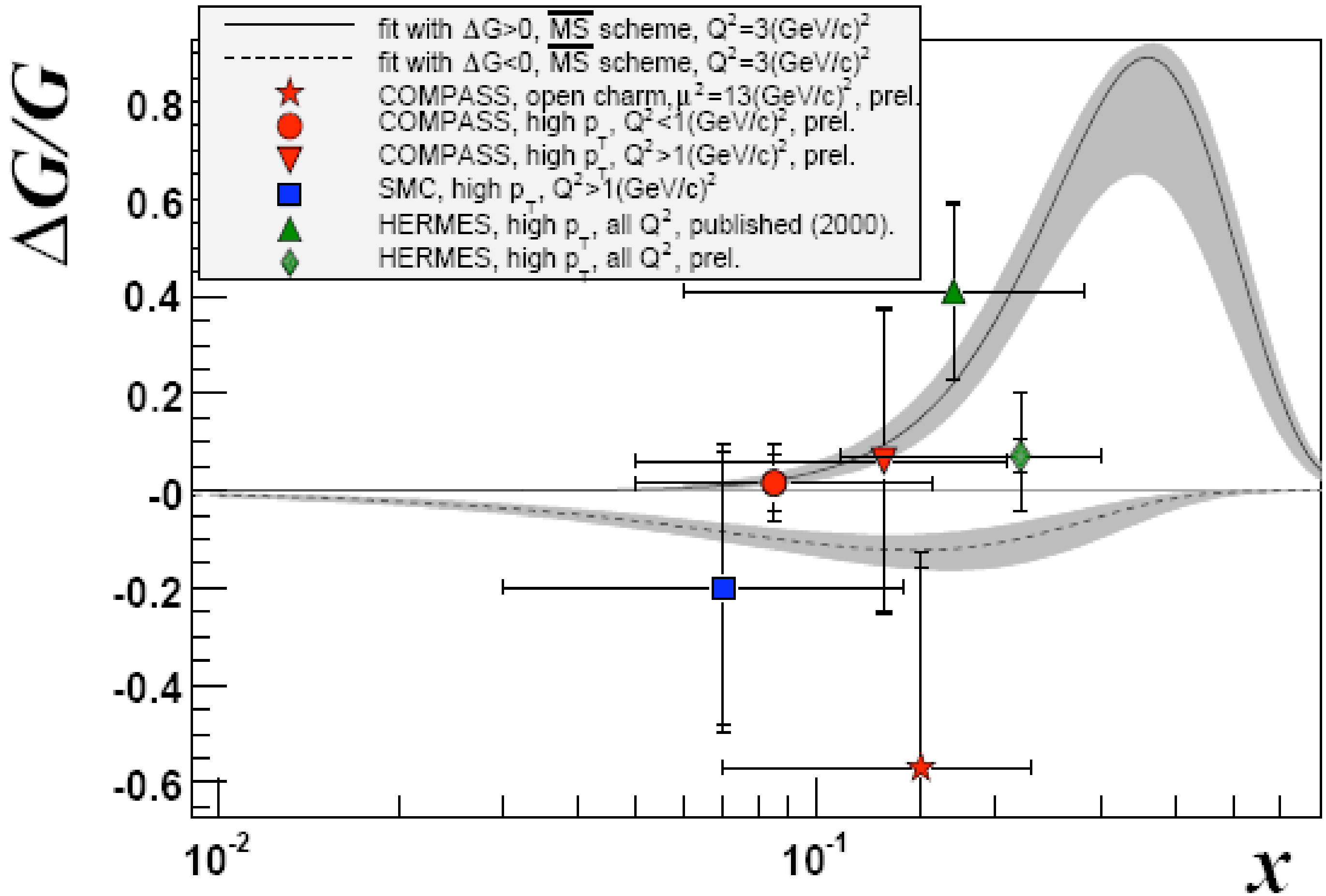


for which no physical explanation was ever offered

This spurred a tremendous experimental effort

- DIS measurements of spin structure functions of polarized p, d, ^{3}He (and ^{6}Li) at SLAC, CERN, Hermes, JLab
- Direct search for high-p_T hadrons at Hermes, COMPASS, RHIC to directly search for effects of polarized glue in the p
- This effort has lasted the past 20 years, with great success

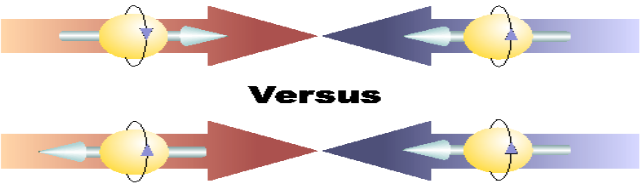
Gluon polarisation



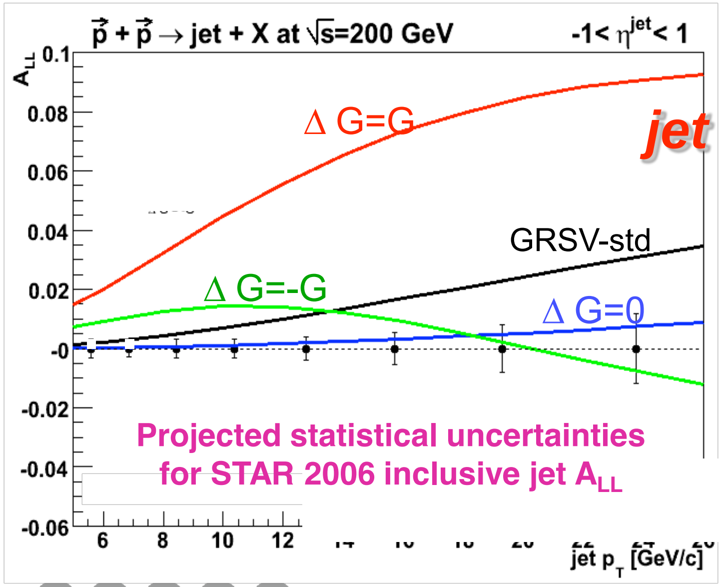
- bands correspond to statistical errors
- uncertainty due to parameterization not included

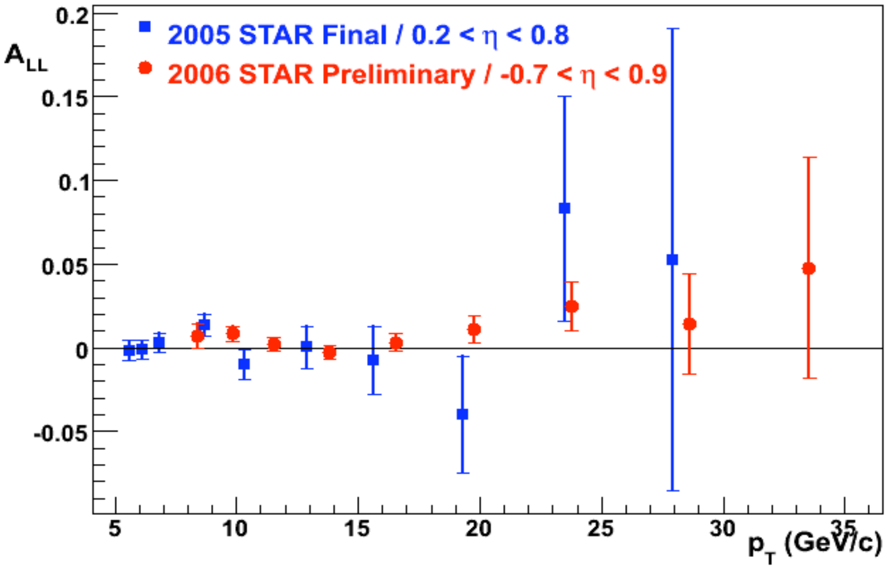
Kabuβ - Pacific-SPIN07

Latest STAR result - Sarsour DNP Oct 07

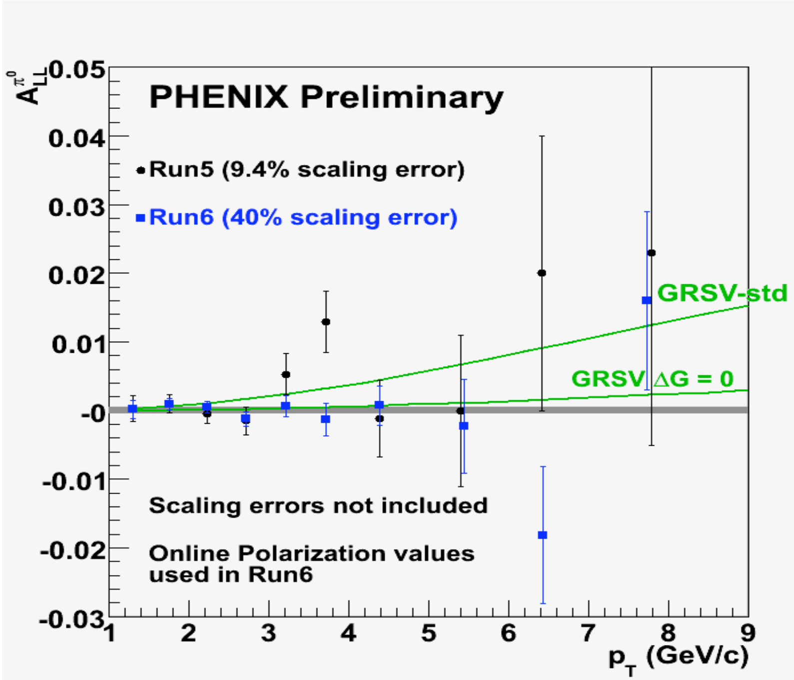


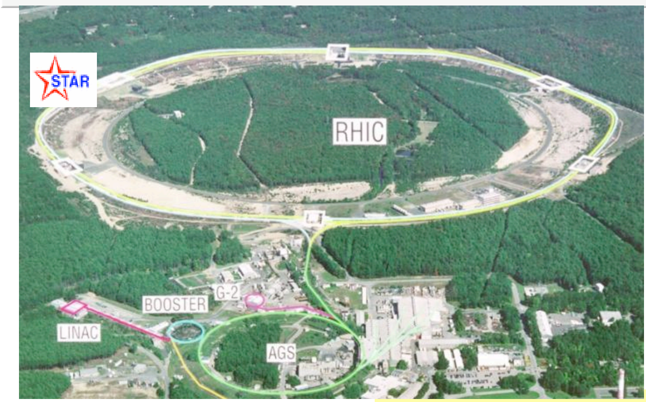
- NLO pQCD describes inclusive jet cross section at RHIC
- Within GRSV framework, 2005 results constrain ΔG to less than 65% of the proton spin with 90% confidence
- Significant increase in precision in Run 2006 data provides even stronger constraints on gluon polarization



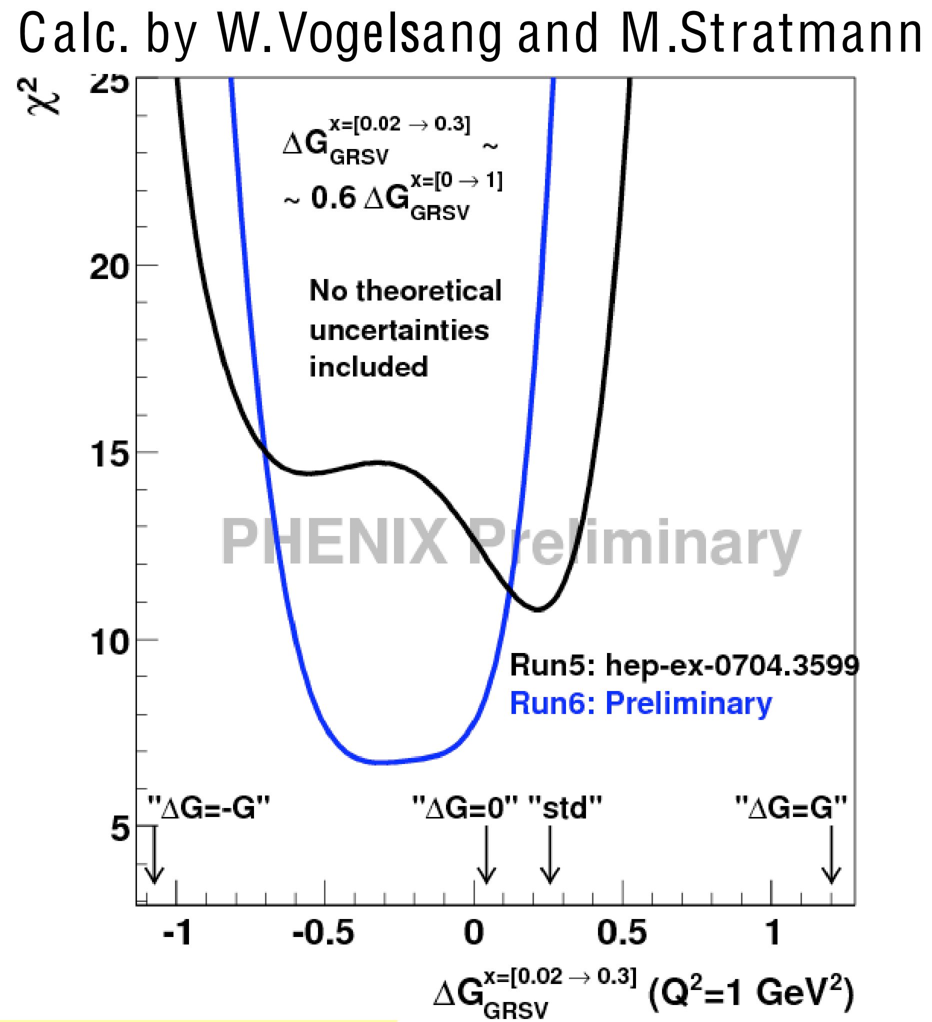


Latest PHENIX Result: From A_{LL} to ΔG





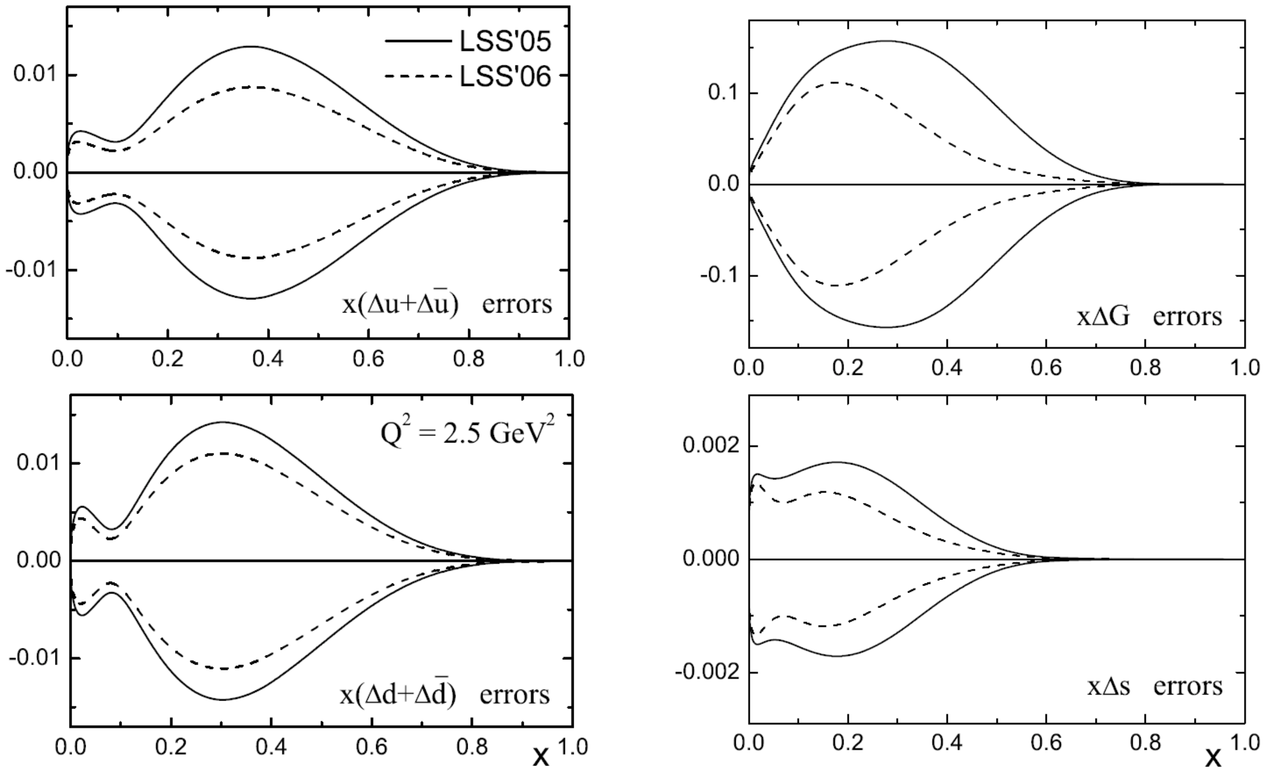
⇒



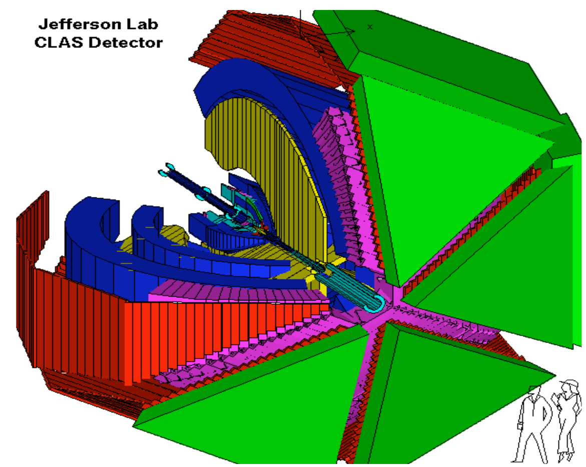
"std" scenario, $\Delta G(Q^2=1GeV^2)=0.4$, is excluded by data on >3 sigma level

Impact of CLAS Precision Data on Parton Distribution Functions

CLAS precision data more than doubled the data points in the DIS region from 30 years of high energy polarized structure function measurements.



At moderate x = 0.4, the relative uncertainty of xΔG is reduced by a factor 3 and of Δs-Δs by a factor 2.

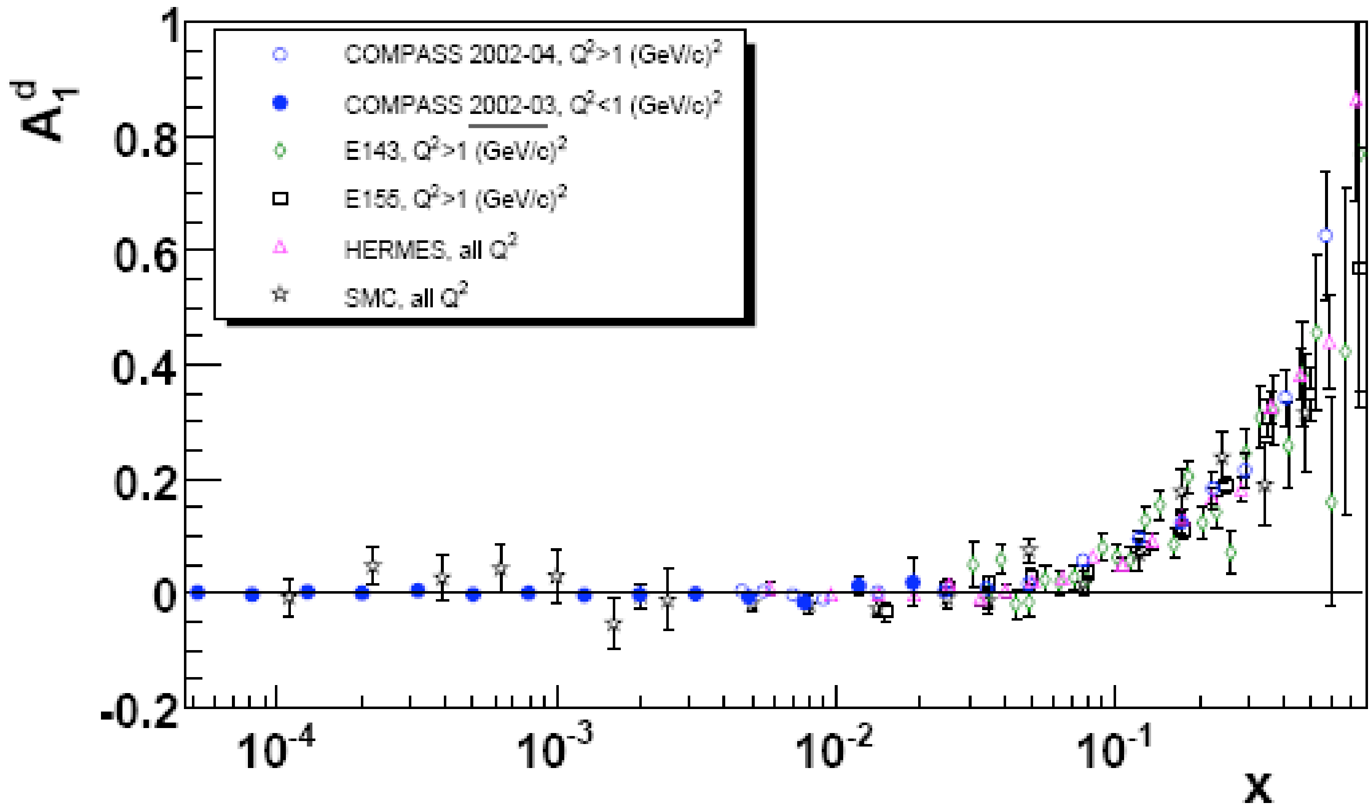


The dashed lines include the CLAS data in the analysis (LSS'06).

E. Leader, A. Sidorov, D. Stamenov, Phys.Rev.D75:074027,2007.

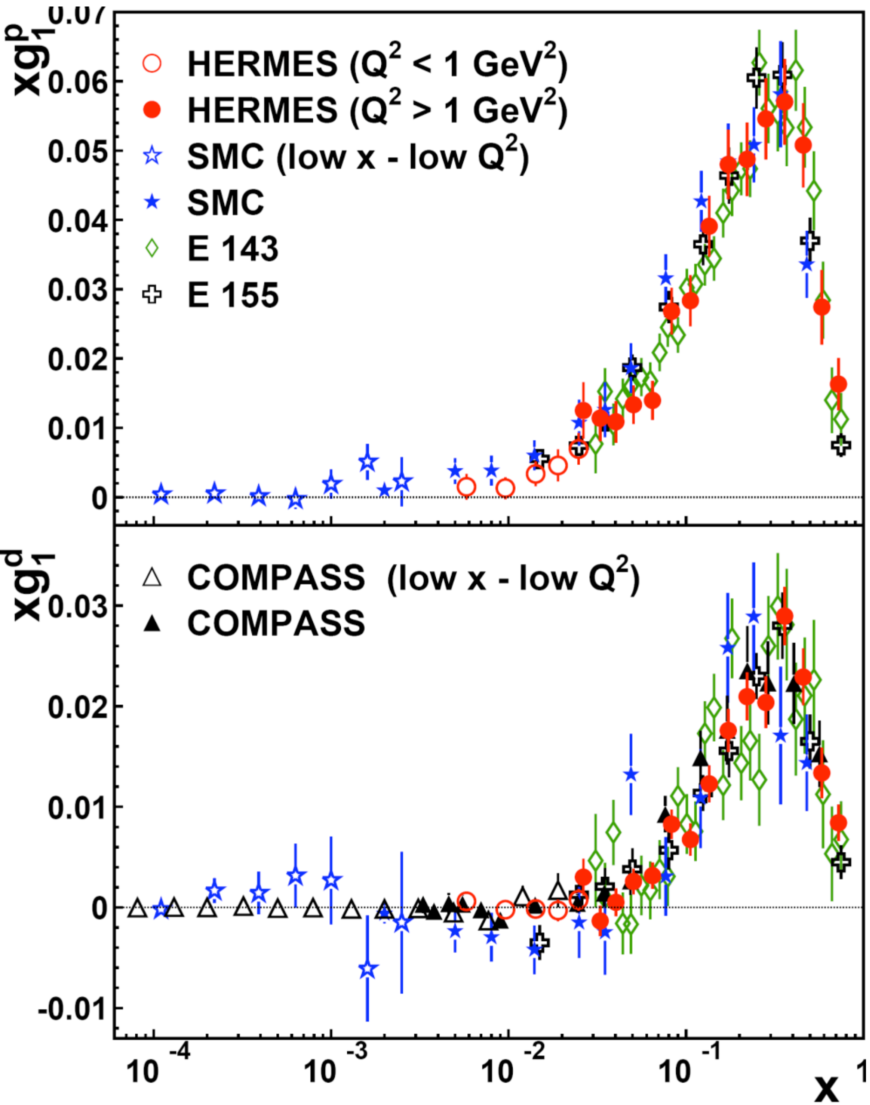
Conclude
$|\Delta G| < 0.3$
at $Q^2 = 1\ GeV^2$

Progress in Determination of $g_1(x)$



- very good agreement with SMC (the only other experiment at low x)
- factor 10–20 improvement of statistical errors compared to SMC

Latest Results for the Integral of g_1



	(theory)	(exp)	(evol)
a_0 = 0.330	0.011	0.025	0.028

Bradamante Erice 0907

$a_0 = 0.33 \quad 0.03^{(stat)} \quad 0.05^{(sys+evol)}$

$\Sigma = a_0$ in $\overline{MS}$

Where is the Spin of the proton?

- Modern data yields:
  $\Sigma = 0.33 \pm 0.03 \pm 0.05$

  (c.f. $0.14 \pm 0.03 \pm 0.10$ originally)

- In addition, there is little or no polarized glue
  - COMPASS: $g^D_1 = 0$ to $x = 10^{-4}$
  - A_{LL} (π^0 and jets) at PHENIX & STAR $\rightarrow \Delta G \sim 0$
  - Hermes, COMPASS and JLab: ΔG small

- Hence: axial anomaly plays little or no role in explaining the spin crisis

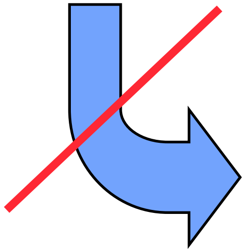
- Return to alternate explanation lost in 1988 in rush to explore the anomaly

Ancient History of the Spin Crisis

- EMC Spin Paper: 22 Dec 87 - 19 May 88
- Brodsky et al. Skyrme: 22 Feb 88 - 19 May 88
- Schreiber-Thomas CBM: 17 May 88 - 8 Dec 88
- Myhrer-Thomas OGE: 13 June 88 - 1 Sept 88

(neither paper could explain reduction to only 14%!)

- Efremov-Teryaev Anomaly: 25 May 88
- Altarelli-Ross Anomaly: 29 June 88 - 29 Sept 88



One-Gluon-Exchange Correction

PHYSICAL REVIEW D VOLUME 38, NUMBER 5 1 SEPTEMBER 1988

Rapid Communications

The Rapid Communications section is intended for the accelerated publication of important new results. Since manuscripts submitted to this section are given priority treatment both in the editorial office and in production, authors should explain in their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than 3½ printed pages and must be accompanied by an abstract. Page proofs are sent to authors, but, because of the accelerated schedule, publication is not delayed for receipt of corrections unless requested by the author or noted by the editor.

Spin structure functions and gluon exchange

F. Myhrer
Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208

A. W. Thomas
*Department of Physics and Mathematical Physics, University of Adelaide, Adelaide, South Australia 5000, Australia and Department of Theoretical Physics, Oxford University, Oxford OX1 3NP, Oxfordshire, England**

(Received 13 June 1988)

Two-quark correlations due to gluon exchange give corrections to both the proton and neutron spin-dependent structure functions in the Bjorken sum rule. They are found to be as large as the pionic corrections in the cloudy bag model of the nucleon. While still not enough to explain the result published recently by the European Muon Collaboration, it is compatible with the reanalysis of the data by Close and Roberts.

PHYSICAL REVIEW D VOLUME 37, NUMBER 7 1 APRIL 1988

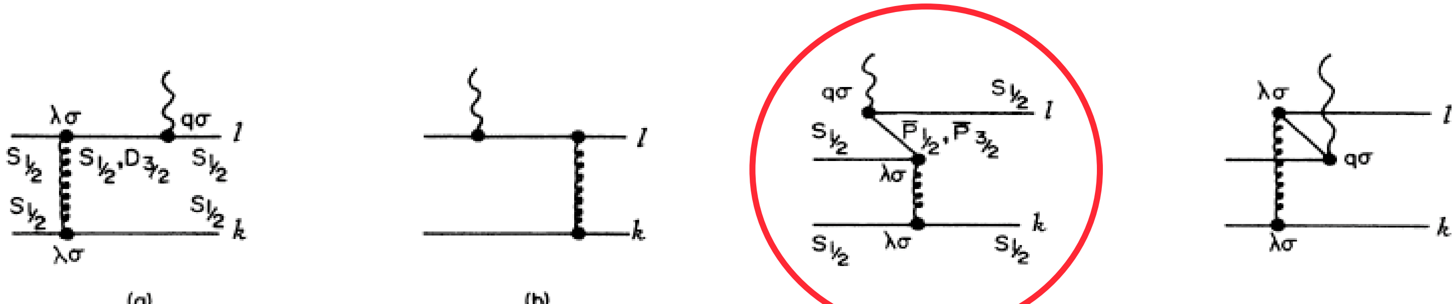
SU(6) violations due to one-gluon exchange

H. Høgaasen
Fysisk Institutt, University of Oslo, Blindern, 0316 Oslo 3, Norway

F. Myhrer
Department of Physics, University of South Carolina, Columbia, South Carolina 29208

(Received 26 October 1987)

The one-gluon-exchange corrections to the baryon magnetic moments and the weak semileptonic decays are shown to have the correct two-body operator in order to explain recent data. An explicit model calculation using a mode sum for the quark propagator is then performed. In this model calculation the two lowest states dominate the corrections. This value of SU(6) breaking explains the measured ratio $\Sigma^- \to ne\bar{\nu}/\Lambda \to pe\bar{\nu}$ as well as why $\mu_{\Xi^-} < \mu_\Lambda$ and it restores $\mu_p/\mu_n \simeq -\frac{3}{2}$ in chiral bag models.



(a) (b) (c) (d)

Intermediate quark state contributing M	Intermediate quark energy	$10^4\Delta\mu$	$10^4\Delta g_A$	Intermediate quark energy M	$10^4\Delta\mu$	$10^4\Delta g_A$
$S'_{1/2}$	$5.40/R$	22	32	$8.58/R$	1.0	2.2
$D_{3/2}$	$5.12/R$	8	12	$8.41/R$	0.4	0.8
$\bar{P}_{1/2}$	$3.81/R$	730	−275	$7.00/R$	−6.7	7.0
$\bar{P}_{3/2}$	$3.20/R$	1349	−332	$6.76/R$	−6.1	6.0
Sum		2109	−563		−11.4	16.0

OGE Correction for Hyperon β-decay

- All correction terms proportional to $G = \alpha_s$ times bag matrix elements

- Very nicely accounts for deviations from SU(3) symmetry

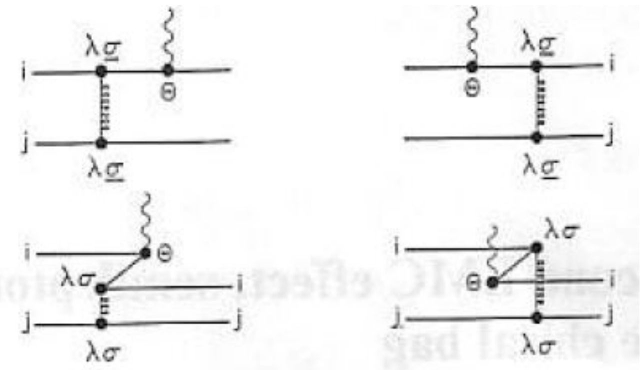


Table 1. The ratio g_A/g_V in the $SU(3)$ limit from a model calculations compared to experiments. The experimental numbers are from the Particle Data Group [32]

	Theory: MIT bag + CMI	$SU(3)$ amplitudes	Experiments
$n \to p$	$\frac{5}{3}B' + G = 1.25$	$F + D$	1.259
$\Sigma^- \to n$	$-\frac{1}{3}B' - 2G = -0.34$	$F - D$	-0.36 ± 0.05
$\Lambda \to p$	$B' = 0.72$	$F + D/3$	0.696 ± 0.025
$\Xi^- \to \Lambda$	$\frac{1}{3}B' - G = 0.19$	$F - D/3$	0.25 ± 0.05

F = 0.45 (fixed)
D = 0.81
D = 0.74
D = 0.60

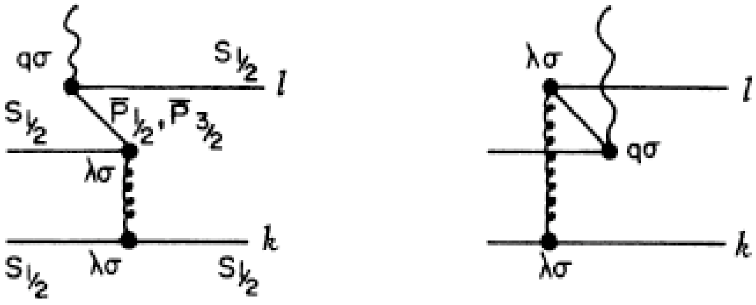
Without OGE correction MIT bag gives $F = 2B'/3$, $D = B'$

Hogaasen & Myhrer, Z. Phys. C48 (1990) 295

One-Gluon-Exchange Correction

- **Has the effect of further reducing the fraction of spin carried by the quarks in the bag model (naively 0.65)**

- **$\Sigma \rightarrow \Sigma - 3G$; with $G \sim 0.05$**
  $\Sigma \rightarrow 0.65 - 0.15 = 0.5$



(c) (d)

- **Effect is to transfer quark spin to quark (relativity) and anti-quark (OGE)** **orbital angular momentum**

The Pion Cloud of the Nucleon
- required by chiral symmetry

Volume 215, number 1 PHYSICS LETTERS B 8 December 1988

SPIN DEPENDENT STRUCTURE FUNCTIONS IN THE CLOUDY BAG MODEL

A.W. SCHREIBER AND A.W. THOMAS
Department of Physics and Mathematical Physics, University of Adelaide, North Terrace, Adelaide, South Australia 5000, Australia

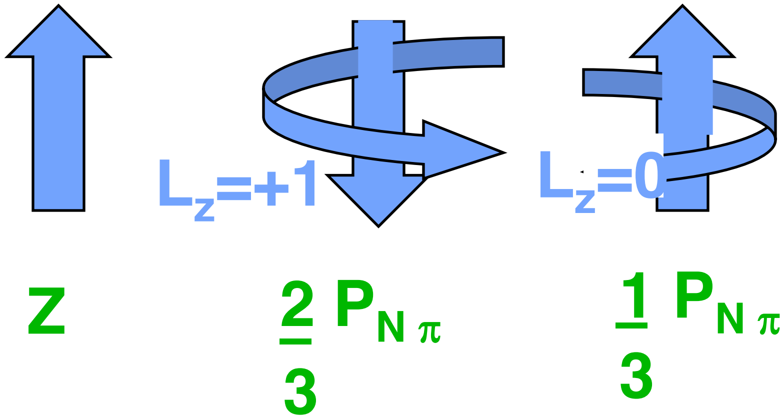
Received 17 May 1988

We derive expressions for the integrals of the spin dependent structure functions $g_1(x)$ for the proton and the neutron in the context of the cloudy bag model. We find that the neutron contributes 5–10% to the Bjorken sum rule, while there is a corresponding decrease for the proton's contribution. It is difficult to reconcile these results with those reported in a recent experiment.

Effect of the Pion Cloud

- Probability to find a bare N is Z ~ 70%

- Biggest Fock Component is $N\pi \sim$ 20-25% and 2/3 of time N spin points down



- Next biggest is $\Delta\pi \sim$ 5-10%

- To this order (i.e. including terms which yield LNA and NLNA contributions):

- Spin gets renormalized by a factor :

$$Z - 1/3\, P_{N\pi} + 15/9\, P_{\Delta\pi} \sim 0.75 - 0.8$$

$$\Rightarrow \Sigma = 0.65 \rightarrow 0.49 - 0.52$$

Support for Pion Cloud Picture

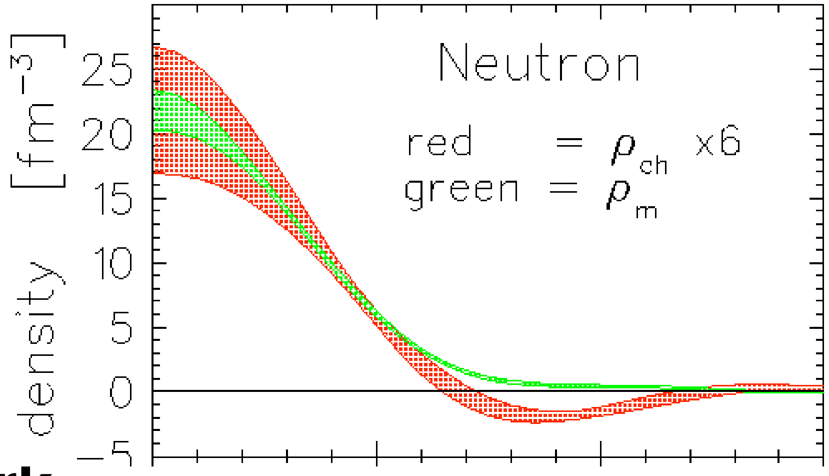
- **Most spectacular example is the prediction*** **of $\bar{d} > \bar{u}$, because of the pion cloud ($p \to n\ \pi^+$)**

$$\int_0^1 dx\ [\ \bar{d} - \bar{u}\ ] = 2\ P_{N\pi}/3 - P_{\Delta\pi}/3 \in 0.11 - 0.15$$

(in excellent agreement with latest data)

J.J. Kelly

* Thomas, Phys. Lett. B126 (1983) 97



- **Charge distribution of the neutron**

- **Natural understanding of quark mass dependence of data from lattice QCD (later)**

Can one add OGE and Pion Corrections?

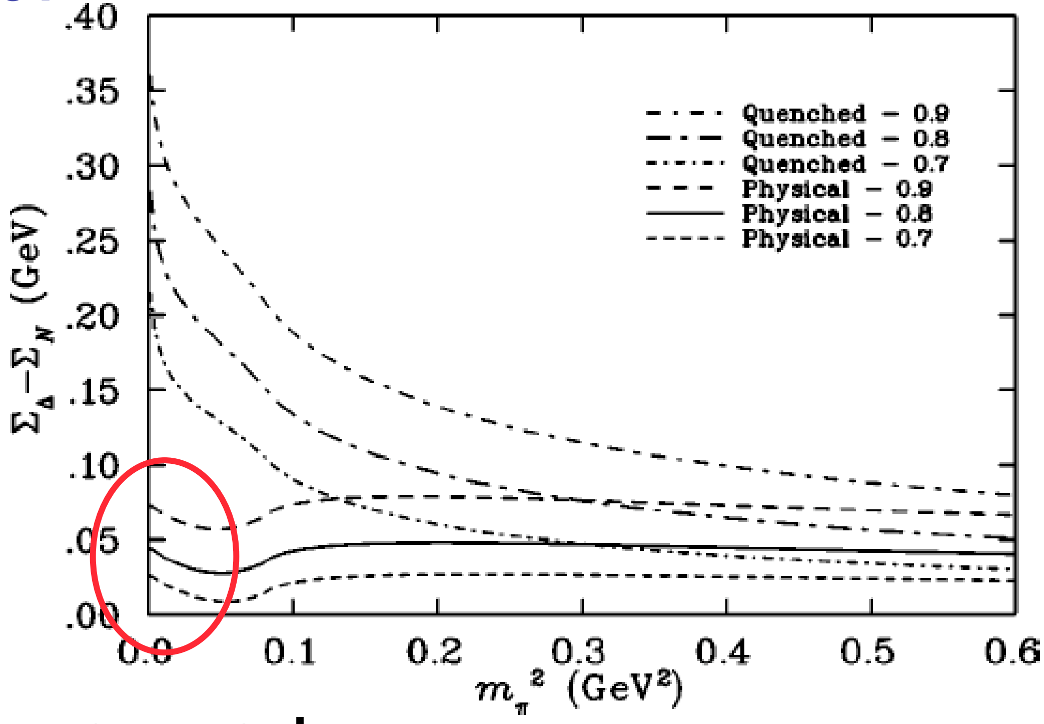
- Prime phenomenological need for OGE interaction is the hyperfine splitting of N and Δ masses, Λ and Σ masses, etc. – i.e. hadron spectroscopy
- In early days of chiral models believed some of this hyperfine splitting came from pion self-energy differences
- Maybe double counting to include correction to Σ from both pions and OGE??
- Modern understanding *NO*: from analysis of data in quenched (QQCD) and full QCD, from Lattice QCD - implies 50 MeV (or less) of $m_\Delta - m_N$ in this way

Young et al., Phys. Rev. D66 (2002) 094507

Nucleon - Δ Splitting

Lattice analysis
⇒ pions give 40 ± 20 MeV

PHYSICAL REVIEW D **66**, 094507 (2002)



- **Hence most of the N-Δ splitting comes from OGE – as in most quark models**

- **Thus the value of α_s used in the bag model calculation of the exchange current correction is more or less unchanged**

- **and… one <u>can add</u> the pion and OGE corrections to the spin sum-rule**

Final Result for Quark Spin

$\Sigma = (Z - P_{N\pi}/3 + 5\, P_{\Delta\pi}/3) \times (0.65 - 3\, G)$

$= (0.7, 0.8) \times (0.65 - 0.15) = (0.35, 0.40)$

c.f. Experiment: $0.33 \pm 0.03 \pm 0.05$

- ALL effects, relativity and OGE and the pion cloud have the effect of swapping quark spin for valence orbital angular momentum and anti-quark orbital angular momentum (>60% of the spin of the proton)

Myhrer & Thomas, hep-ph/0709.4067

The Balance Sheet – fraction of total spin

	L_{u+ubar}	L_{d+dbar}	Σ
Non-relativistic			1.0
Relativity (e.g. Bag)	0.46	-0.11	0.65
Plus OGE (-0.15)	0.67	-0.16	0.49
Plus pion ($\times$ 0.8)	0.64	-0.03	0.39

At model scale: $L_u + S_u = 0.32 + 0.42 = 0.74 = J_u$

: $L_d + S_d = -0.02 - 0.22 = -0.24 = J_d$

LHPC Lattice Study

- At first glance shocking : $L^u \sim -0.1$ and $L^d \sim +0.1$
  (c.f. + 0.32 and - 0.02 in our "resolution")
- N.B. Disconnected terms missing $\rightarrow$ no anomaly, sea wrong

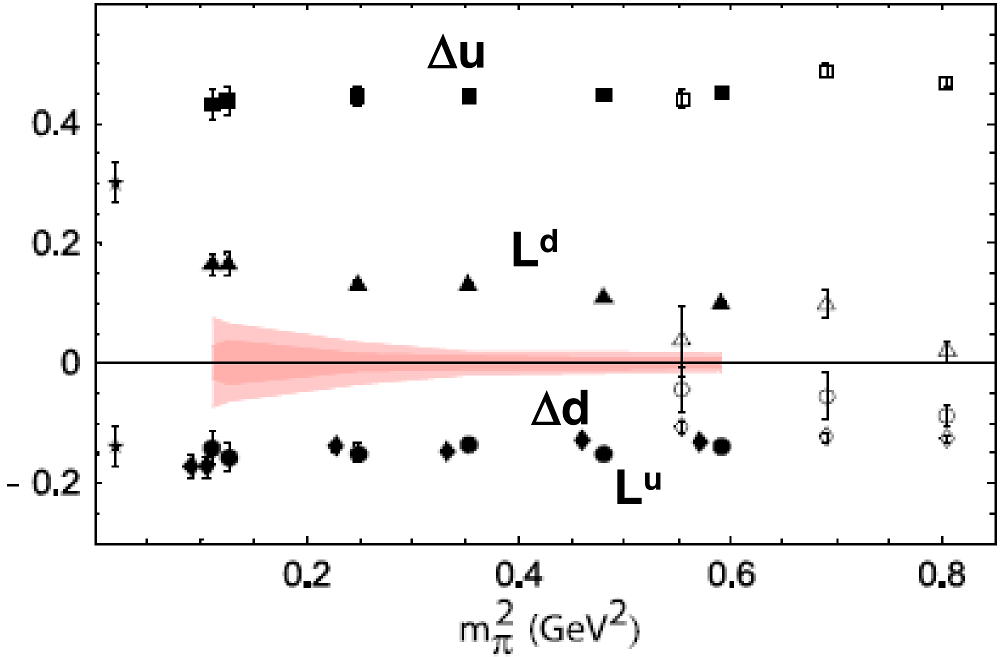


Figure 16: Nucleon spin decomposition by flavor. Squares denote $\Delta\Sigma^u/2$, diamonds denote $\Delta\Sigma^d/2$, triangles denote L^u, and circles denote L^d.

LHPC: hep-lat/0610007

The Balance Sheet – fraction of total spin

	L_{u+ubar}	L_{d+dbar}	Σ
Non-relativistic			1.0
Relativity	0.46	- 0.11	0.65
Plus OGE	0.67	- 0.16	0.49
Plus pion	0.64	- 0.03	0.39

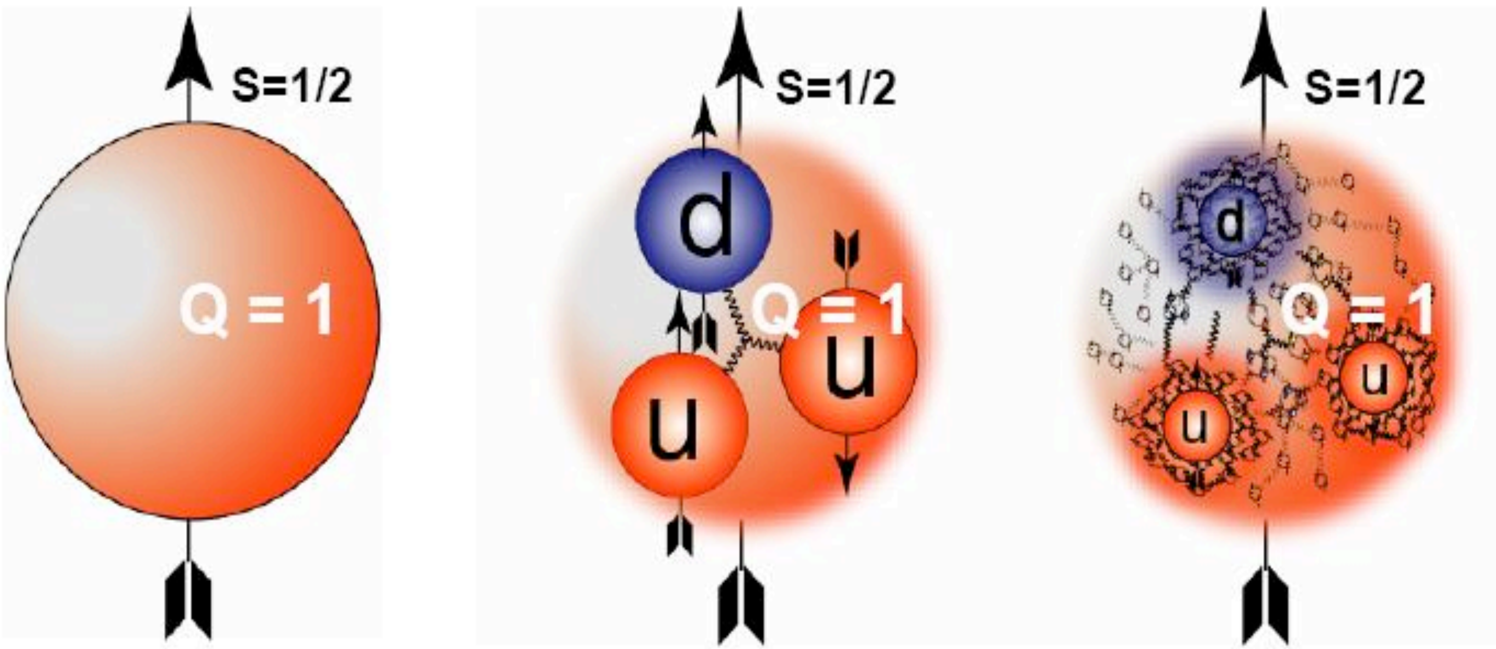
At model scale: $L_u + S_u = 0.32 + 0.42 = 0.74 = J_u$

: $L_d + S_d = - 0.02 - 0.22 = - 0.24 = J_d$

Solution

What we "see" changes with spatial resolution

>1 fm	0.1 — 1 fm	< 0.1 fm
Nucleons	Constituent quarks and glue	"bare" quarks and glue



Indeed L_z is not scale invariant – what scale?

- Known since mid-70s (Le Yaouanc et al., Parisi, etc.) that connection between quark models and QCD must be at low-Q^2

- This is because momentum fraction carried by quarks is monotonically decreasing with $Q^2 \uparrow$ and in models quarks carry nearly all the momentum (used by Glück-Reya to model HERA data to very low x - $\mu^2 = 0.23$ GeV^2 at LO – Phys Lett 359, 205 (1995))

e.g. Schreiber et al., PR D42, 2226 (1990) : $\mu = 0.5$ GeV

(N.B. Using LO rather than NLO QCD changes μ not the results at 5-10 GeV^2)

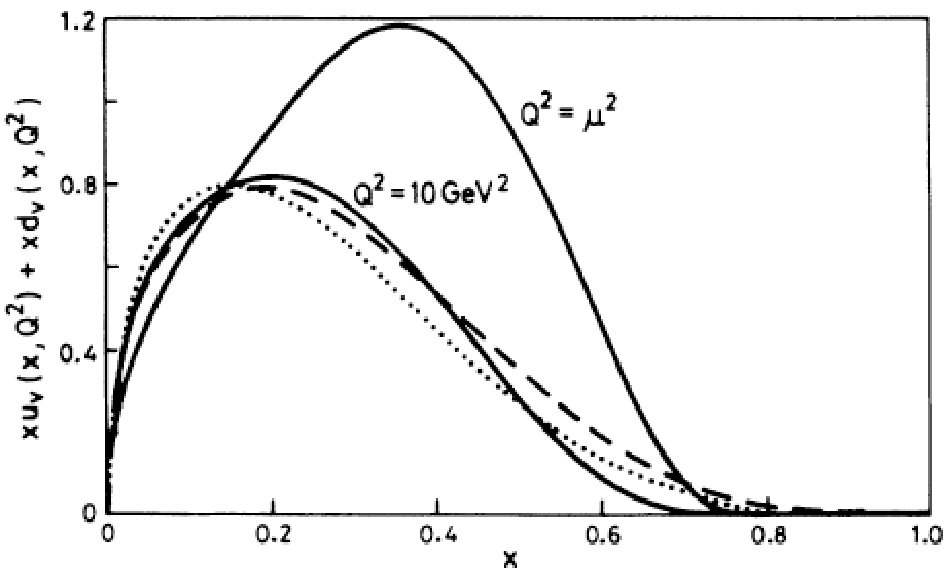
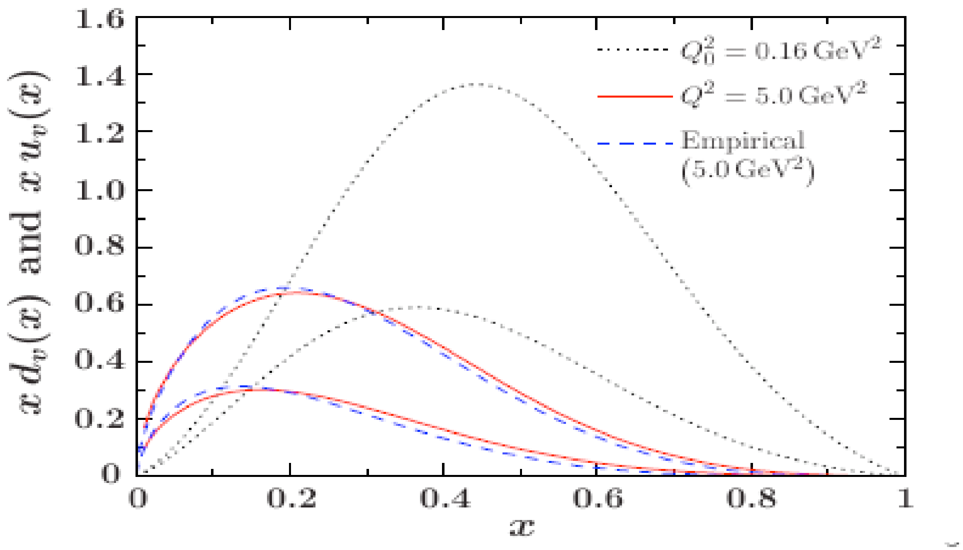
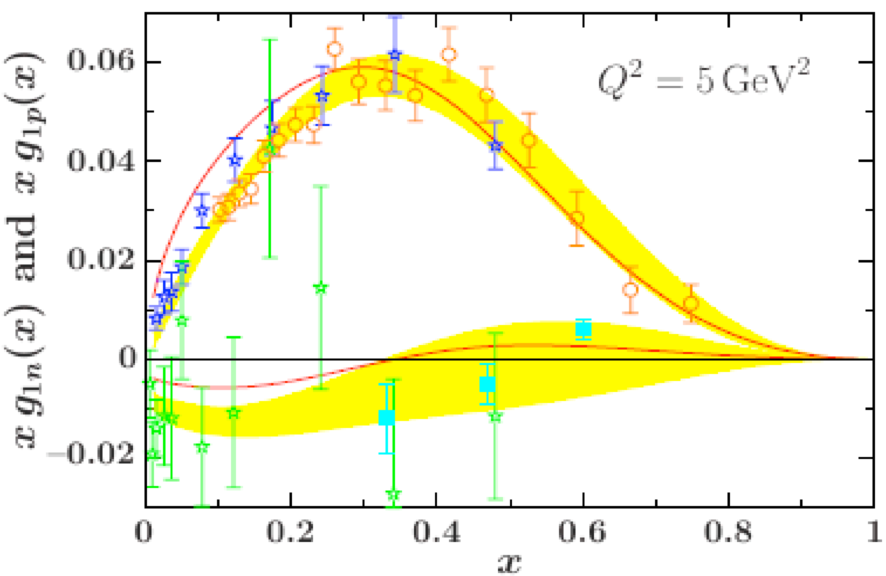


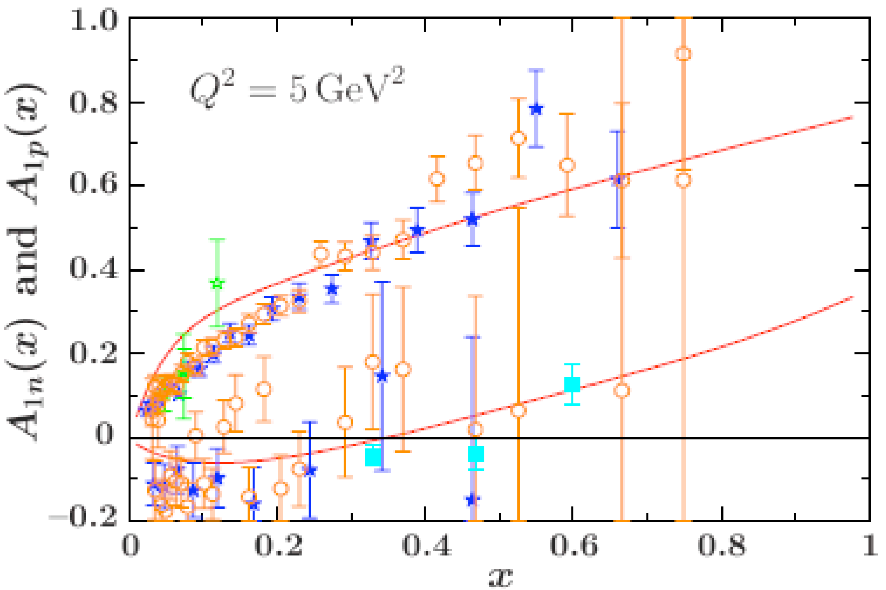
FIG. 1. $xu_v(x,Q^2)+xd_v(x,Q^2)$ at the model scale $Q^2=\mu^2$ and at $Q^2=10$ GeV^2 (solid lines). The dashed and dotted lines correspond to the Duke-Owens and Martin-Roberts-Stirling parametrizations of $xu_v(x,Q^2=10\text{ GeV}^2)+xd_v(x,Q^2=10\text{ GeV}^2)$,

More Modern (Confining) NJL Calculations

Cloet et al.,
Phys. Lett. B621, 246 (2005)
(μ = 0.4 GeV)

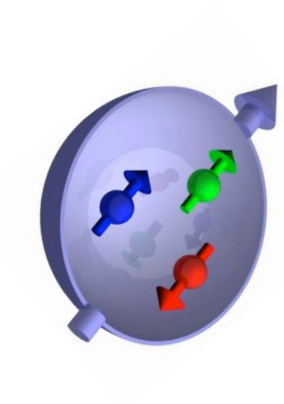


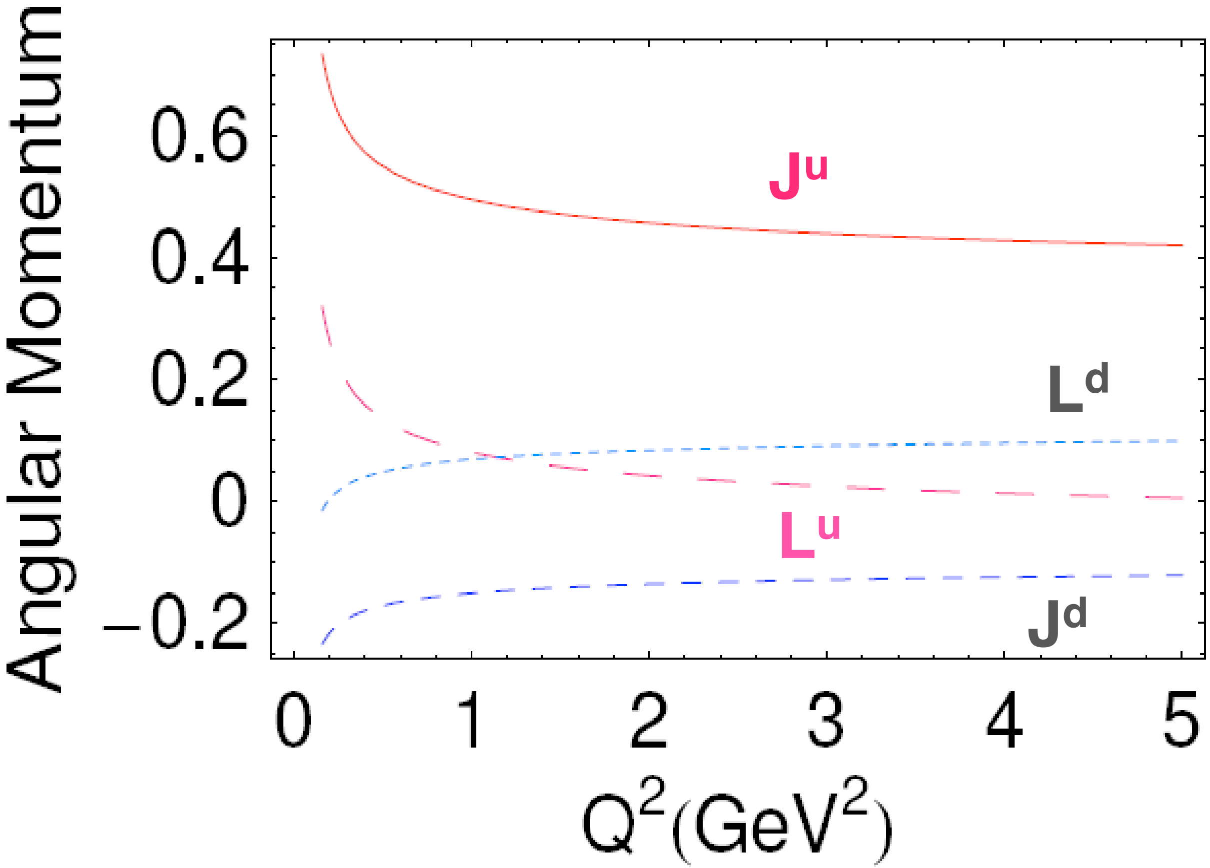


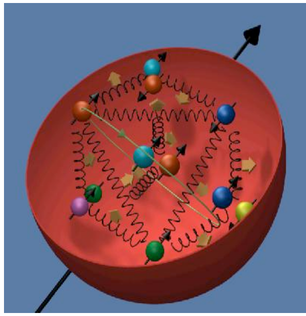


Solution of the Evolution Equations

L^u and L^d both small and cross-over rapidly: AWT hep-ph/0803.2775





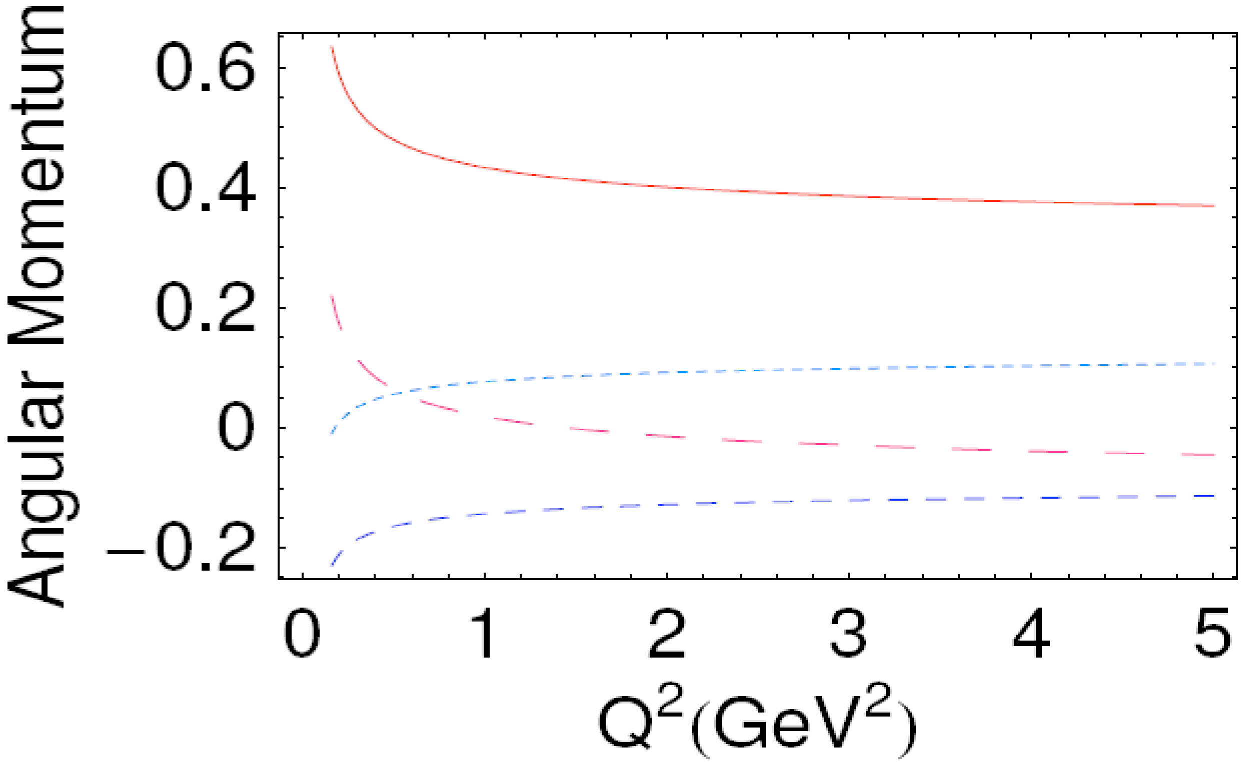


$\Delta G = 0, \Sigma = 0.39, L^u = 0.33, L^d = -0.02, L^g = 0, Q_0 = 0.4$ GeV

$\Rightarrow L^u = +0.01 , L^d = +0.10 , J^u = +0.43, J^d = -0.12$ at 4 GeV^2

Effect of Polarized Glue – or Gluon Angular Momentum

N.B. Evolution for quarks does not distinguish ΔG from L^g

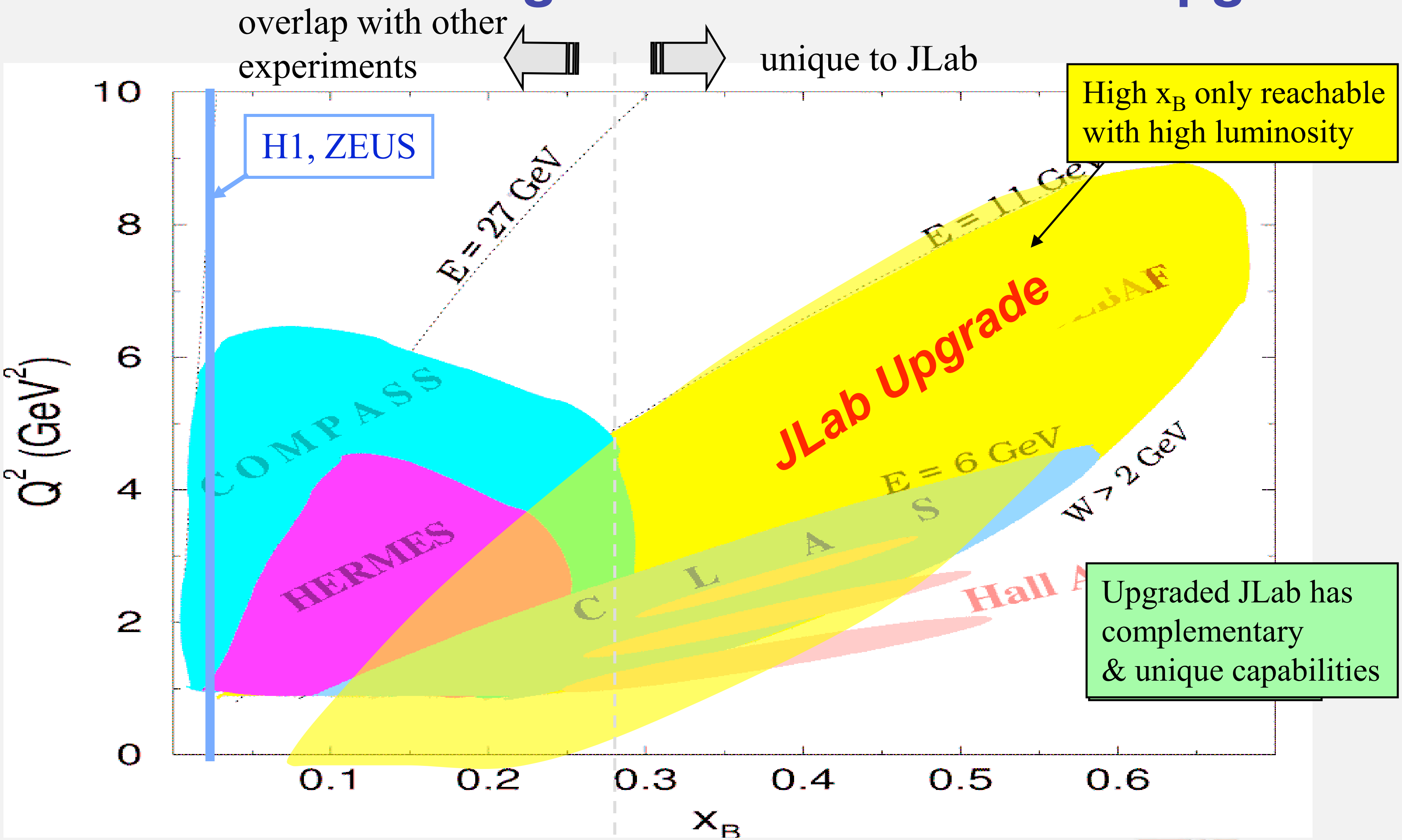


$\Delta G = 0.1$, $\Sigma = 0.39$, $L^u = 0.21$, $L^d = -0.01$, $L^g = 0$, $Q_0 = 0.4$ GeV

$\Rightarrow L^u = -0.04$, $L^d = +0.10$, $J^u = +0.38$, $J^d = -0.12$ at 4 GeV^2

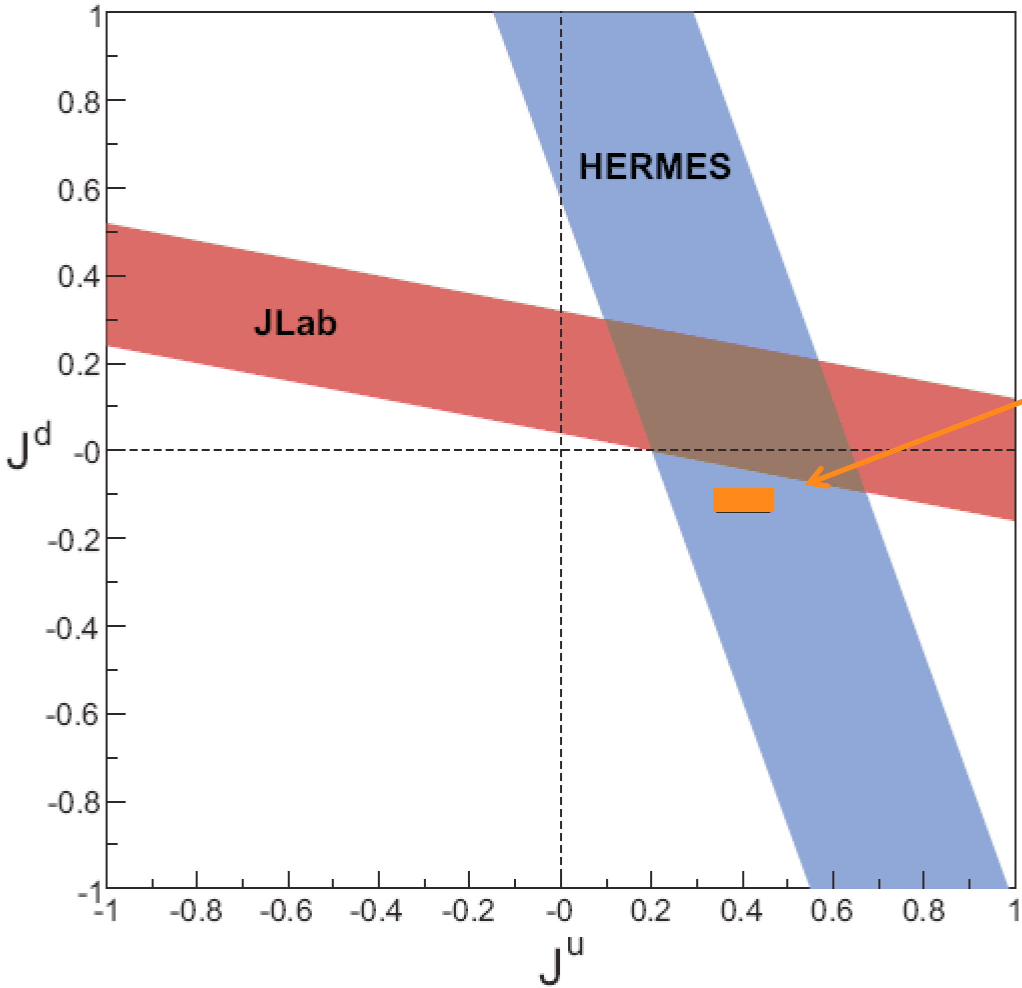
c.f. (+.01) (+0.10) (+0.43) (-0.12)

Deeply Virtual Exclusive Processes - Kinematics Coverage of the JLab 12 GeV Upgrade



Experimental Constraints: Already at 6 GeV

Within model of Vanderhaeghen and collaborators…. model dependence?



Comparison with model of Myhrer & Thomas

(from: hep-ph/0803.2775)

Mazouz et al. (JLab), PRL 99, 242501 (2007)

Summary

- Two decades of experiments have given us important new insight into spin structure of the p

- U(1) axial anomaly appears to play little role in resolving the problem
  - not as severe as in original EMC paper

- Instead, important details of the non-perturbative structure of the nucleon DO resolve the "crisis"
  - OGE hyperfine interaction
  - chiral symmetry: pion cloud
  - relativistic motion of quarks

Ingredients of a minimal description of proton structure

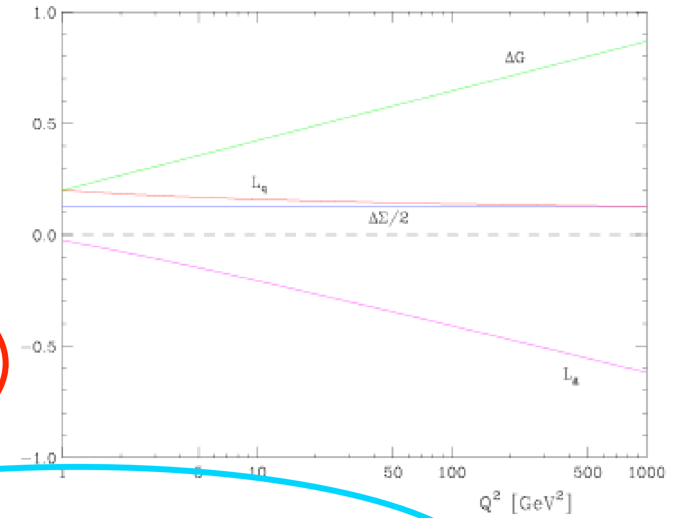
Summary

- Important consequence for quark model: significant orbital angular momentum carried by valence quarks and anti-quarks in the proton

- Effect of QCD Evolution is to:
  - flip ordering of L^u and L^d
  - severely reduce the magnitude of orbital angular momentum
  - restore agreement between data, LQCD and Myhrer-Thomas explanation of the spin crisis

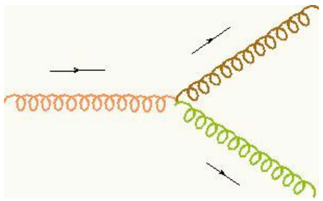
- Study of GPDs at JLab provide the primary tool to verify this (maybe transversity too?)

Evolution Equations - singlet



$$\Delta\Sigma(t) = \text{const},$$

$$\Delta g(t) = -\frac{4\Delta\Sigma}{\beta_0} + \frac{t}{t_0}\left(\Delta g_0 + \frac{4\Delta\Sigma}{\beta_0}\right)$$



$$L_q(t) = -\frac{1}{2}\Delta\Sigma + \frac{1}{2}\frac{3n_f}{16+3n_f} + (t/t_0)^{-2(16+3n_f)/9\beta_0} \times \left(L_q(0) + \frac{1}{2}\Delta\Sigma - \frac{1}{2}\frac{3n_f}{16+3n_f}\right),$$

$$\frac{d}{dt}\begin{pmatrix} L_q \\ L_g \end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix} -\frac{4}{3}C_F & \frac{n_f}{3} \\ \frac{4}{3}C_F & -\frac{n_f}{3} \end{pmatrix}\begin{pmatrix} L_q \\ L_g \end{pmatrix} + \frac{\alpha_s(t)}{2\pi}\begin{pmatrix} -\frac{2}{3}C_F & \frac{n_f}{3} \\ -\frac{5}{6}C_F & -\frac{11}{2} \end{pmatrix}\begin{pmatrix} \Delta\Sigma \\ \Delta g \end{pmatrix}$$

$$L_g(t) = -\Delta g(t) + \frac{1}{2}\frac{16}{16+3n_f} + (t/t_0)^{-2(16+3n_f)/9\beta_0} \times \left(L_g(0) + \Delta g(0) - \frac{1}{2}\frac{16}{16+3n_f}\right).$$

Ji, Tang, Hoodbhoy: PRL 76 (1996) 740
Earlier Ratcliffe, Phys Lett B192 (1987)

Non-singlet Equations for Individual Flavors

$$L^{u-d}(t) + \frac{\Delta u - \Delta d}{2} = \left(\frac{t}{t_0}\right)^{-\frac{32}{9\beta_0}} \left(L^{u-d}(t_0) + \frac{\Delta u - \Delta d}{2}\right)$$

Also solve for non-singlet: $L^{u+d} - 2L^s$

$\Rightarrow$

$$\begin{aligned} L^{u(d)} &= -\frac{\Delta u}{2}\left(-\frac{\Delta d}{2}\right) + 0.06 \\ &+ \frac{1}{3}\left(\frac{t}{t_0}\right)^{-\frac{50}{81}} \left[L^{u+d}(t_0) + \frac{\Sigma}{2} - 0.18\right] \\ &+ \frac{1}{6}\left(\frac{t}{t_0}\right)^{-\frac{32}{81}} \left[L^{u+d}(t_0) \pm 3L^{u-d}(t_0) \pm g_A^{(3)} + \frac{\Sigma}{2}\right] \end{aligned}$$