UNIVERSITY OF ICELAND

Stockholm University

Autocorrelators in models with Lifshitz scaling

Lárus Thorlacius

based on
V. Keränen & L.T.: *Class. Quantum Grav. 29 (2012) 194009*
arXiv:1510.nnnn (to appear...)

International workshop on holography and condensed matter systems
University of Perugia, September 24 - 25, 2015

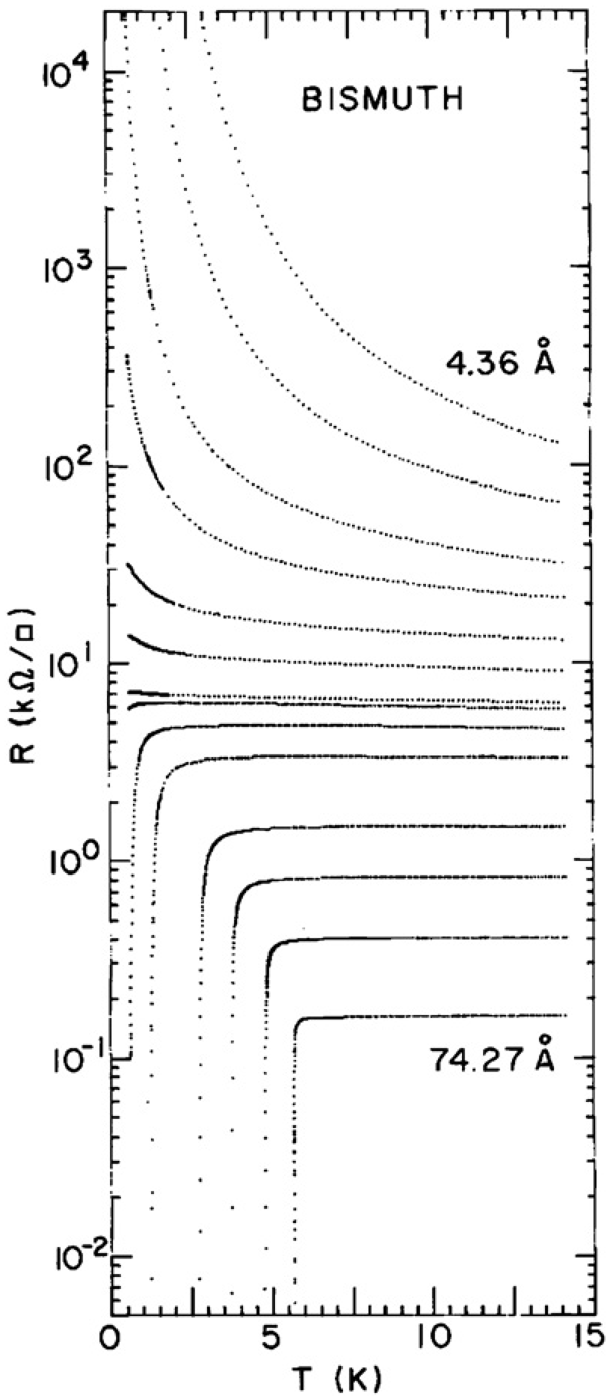
Applied AdS-CFT

Investigate strongly coupled quantum field theories via classical gravity

- growing list of applications:

- hydrodynamics of quark gluon plasma
- jet quenching in heavy ion collisions
- quantum critical systems
  - strongly correlated electron systems
  - cold atomic gases
- out of equilibrium dynamics
- ....

Bottom-up approach: Look for interesting behavior in simple models



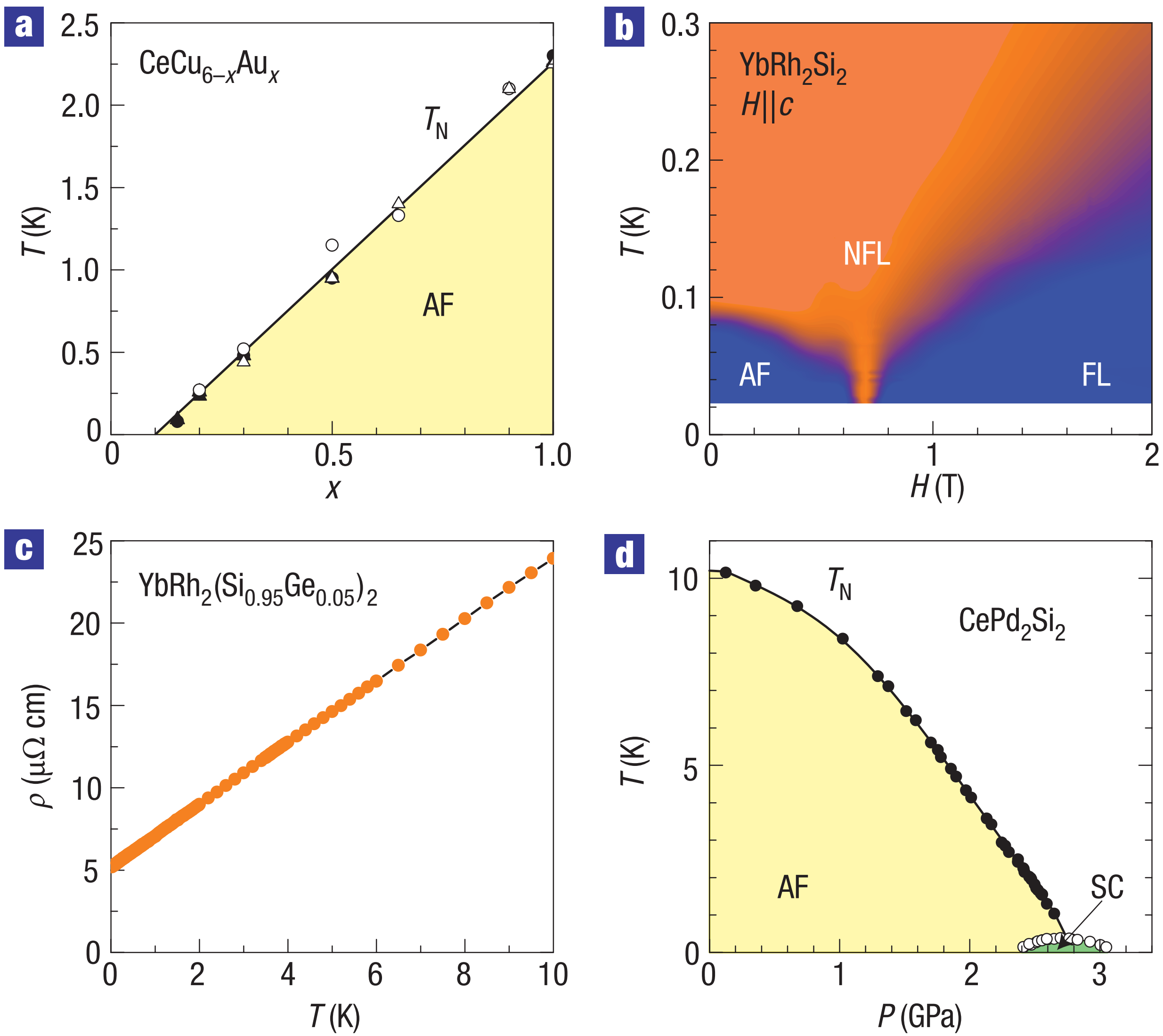
Classic example of a QCP

Resistivity vs. temperature in thin films of bismuth

T = 0 state changes from insulating to superconducting at a critical thickness

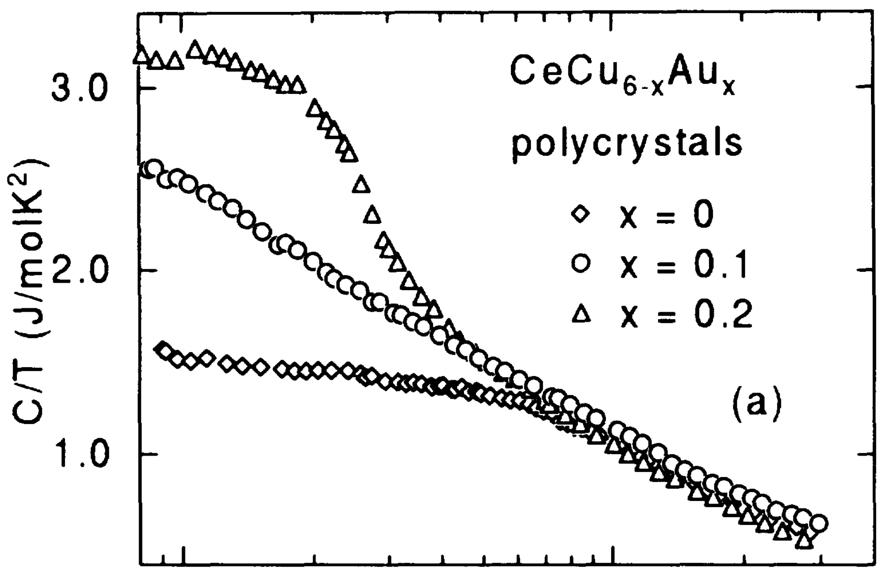
From D.B. Haviland, Y. Liu and A.M. Goldman, Phys. Rev. Lett. **62** (1989) 2180.

Quantum criticality in heavy fermion materials

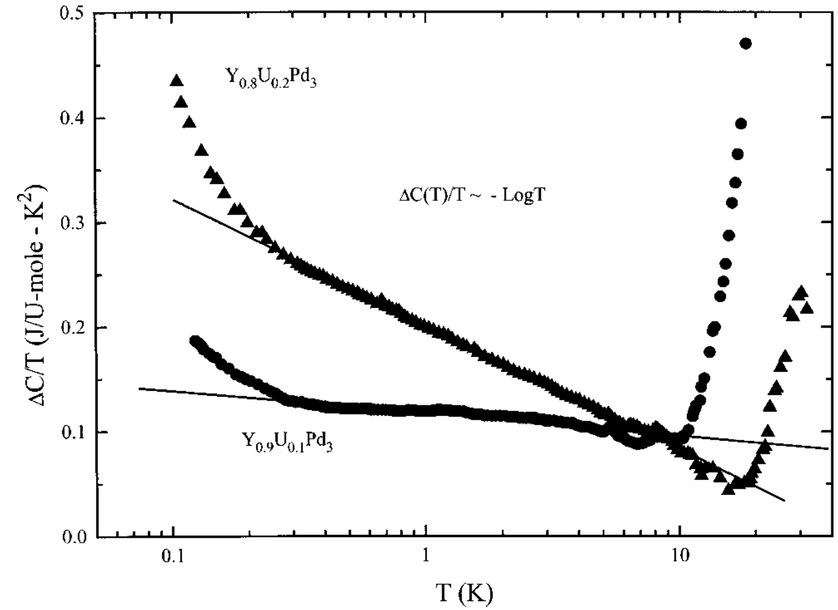


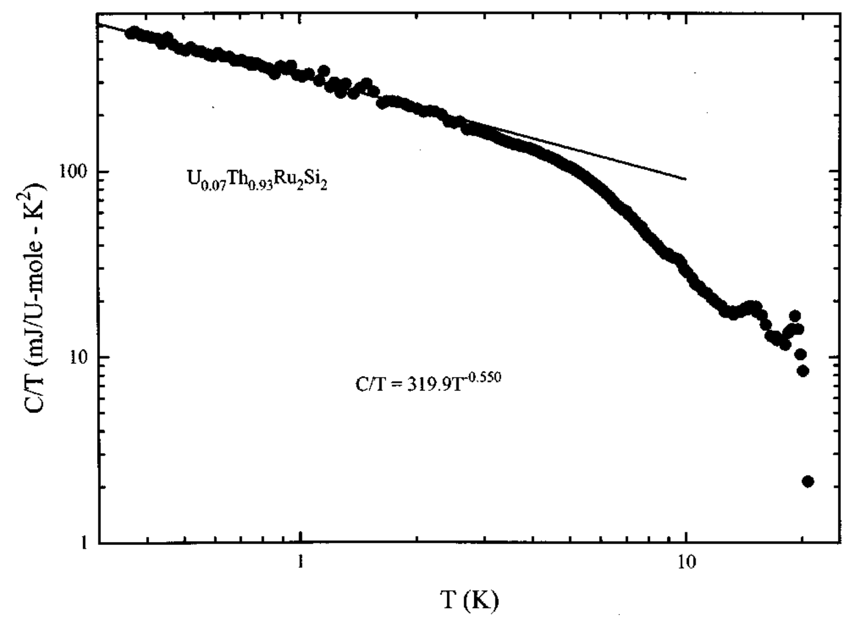
From P. Gegenwart, Q. Si and F. Steglich, Nature Phys. **4** (2008) 186.

Some measured c/T values in heavy fermion metals



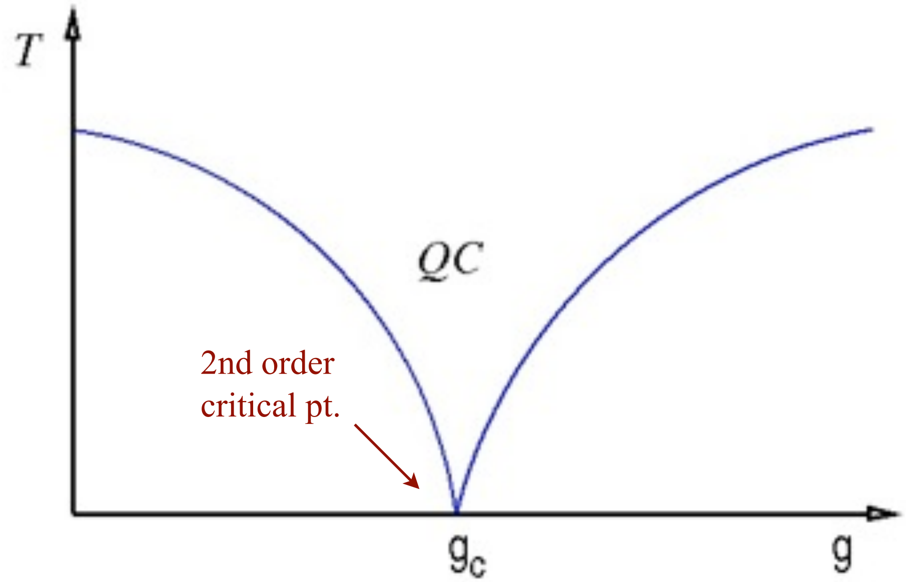
H. Lohneysen et al. *PRL* **72** (1994) 3262.





From G.Stewart, *Rev.Mod.Phys*. **73** (2001) 797.

Quantum critical points



Typical behavior at $T = 0$

characteristic energy $\delta \sim (g - g_c)^{z\nu}$

coherence length $\xi \sim (g - g_c)^{-\nu}$

$\delta \sim \xi^{-z}$ z = dynamical scaling exponent

Scale invariant theory at finite T : $\xi = c\,T^{-1/z}$

Deformation away from fixed pt.: $\lambda_i \sim (\mathrm{length})^{-1}$ QCP has $\lambda_i = 0$

Quantum critical region : $\xi = T^{-1/z}\,\eta(T^{-1/z}\lambda_i)$ $\eta(0) = c$

Physical systems with z = 1, 2, and 3 are known -- non-integer values of z are also possible

$z = 1$ scaling symmetry is part of SO(d+1,1) conformal group = isometries of adS_{d+1}

$z > 1$ scale invariance without conformal invariance - asymptotically Lifshitz spacetime

Models with Lifshitz scaling

$$t \to \lambda^z t, \qquad \mathbf{x} \to \lambda \mathbf{x}, \qquad z \geq 1 \qquad \mathbf{x} = (x_1, \ldots, x_d)$$

Example: Quantum Lifshitz model

$$S = \int d^2x dt \left((\partial_t \phi)^2 - K \left(\nabla^2 \phi \right)^2 \right)$$

$z = 2, \ d = 2$

Q: Can we give a gravity dual description of a strongly coupled system which exhibits anisotropic scaling?

A: Look for a gravity theory in *d +2* dim's with spacetime metric of the form

$$ds^2 = L^2 \left(-r^{2z} dt^2 + r^2 d^2\mathbf{x} + \frac{dr^2}{r^2} \right)$$

$L = 1$

which is invariant under

$$t \to \lambda^z t, \quad \mathbf{x} \to \lambda \mathbf{x}, \quad r \to \frac{r}{\lambda}$$

Kachru, Liu, & Mulligan '08

Holographic models with Lifshitz scaling

Einstein-Maxwell-Proca model

Kachru, Liu, & Mulligan '08; Taylor '08; Brynjólfsson et al. '09

$$S_{\mathrm{EMP}} = \int \mathrm{d}^{d+2}x\sqrt{-g}\,\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{c^2}{2}\mathcal{A}_\mu\mathcal{A}^\mu\right)$$

Einstein-Maxwell-Dilaton model

Taylor '08; Tarrío & Vandoren '11

$$S_{\mathrm{EMD}} = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\sum_{i=1}^{N} e^{\lambda_i\phi}F_i^2\right)$$

we will take N = 1 or 2

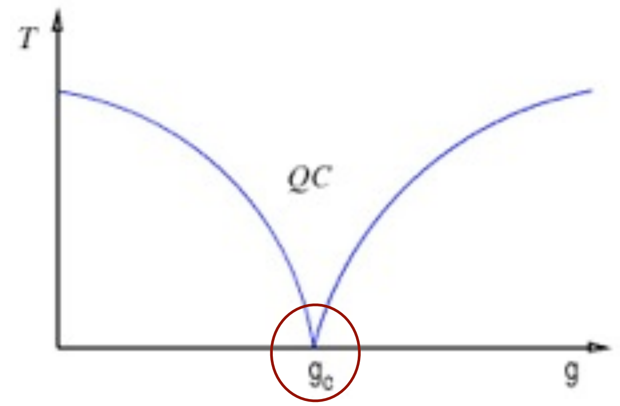
d = 3, 2, or 1 for CM applications

Fixed point metric

The Lifshitz metric

$\mathbf{x} = (x_1, \ldots, x_d)$

$$ds^2 = -r^{2z}dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2}$$



is a solution of both models for particular values of couplings and background fields

EMP model: $$c = \sqrt{z\,d}, \qquad \Lambda = -\frac{z^2 + (d-1)z + d^2}{2}$$

$$\mathcal{A}_t = \sqrt{\frac{2(z-1)}{z}}\, r^z, \quad \mathcal{A}_{x_i} = \mathcal{A}_r = 0 \qquad A_\mu = 0$$

EMD model: $$\lambda_1 = -\sqrt{\frac{2d}{z-1}}\,, \qquad \Lambda = -\frac{(d+z)(d+z-1)}{2}\,, \qquad e^\phi = \left(\frac{r}{r_0}\right)^{\sqrt{2d(z-1)}}$$

$$F^{(1)}_{rt} = 2r_0^{z-1}\sqrt{(d+z)(z-1)}\left(\frac{r}{r_0}\right)^{d+z-1}, \qquad F^{(2)}_{\mu\nu} = 0$$

Gravity duals at finite temperature

periodic Euclidean time: $\tau \simeq \tau + \beta, \quad \beta = \frac{1}{T}$

β introduces an energy scale: scale symmetry is broken

thermal state in field theory: black hole with $T_{\mathrm{Hawking}} = T_{\mathrm{qft}}$

finite charge density in dual field theory: electric charge on BH

magnetic effects in dual field theory: dyonic BH

$z = 1$: AdS-Reissner-Nordström BH in d+2 dimensions

$z > 1$: charged BH in d+2 dimensional $z > 1$ gravity model

Black brane solutions in EMD model $(d = 2)$

Tarrío & Vandoren '11

$$ds^2 = -r^{2z} b(r) dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2 b(r)}$$

$$b(r) = 1 - \left(1 + \frac{\tilde{\rho}^2}{4z}\right)\left(\frac{r_0}{r}\right)^{z+2} + \frac{\tilde{\rho}^2}{4z}\left(\frac{r_0}{r}\right)^{2z+2}$$

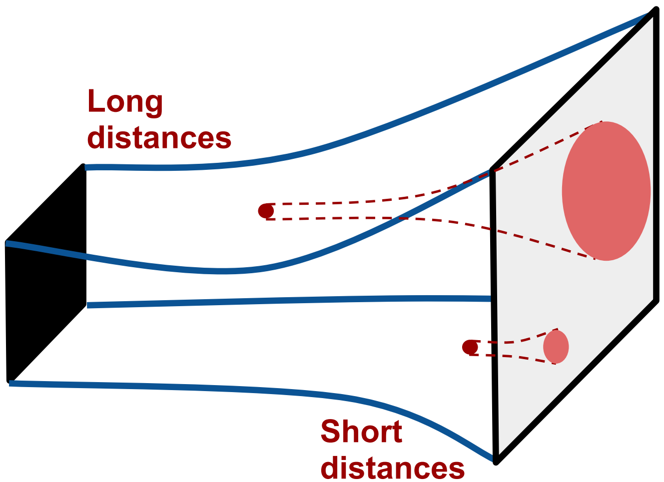
event horizon at $r = r_0$

$$e^{\phi} = \left(\frac{r}{r_0}\right)^{2\sqrt{z-1}}$$

$$F_{rt}^{(1)} = 2 r_0^{z-1} \sqrt{(z+2)(z-1)} \left(\frac{r}{r_0}\right)^{z+1}$$

$$F_{rt}^{(2)} = z \tilde{\rho}\, r_0^{z-1} \left(\frac{r_0}{r}\right)^{z+1}$$

electric charge density



(from S. Hartnoll, arXiv:1106.4342)

Hawking temperature: $T = \frac{r_0^z}{4\pi}\left(z + 2 - \frac{\tilde{\rho}^2}{4}\right)$

AdS-RN black hole: $z \to 1$

Quantum Lifshitz model and 1+1 CFT

Euclidean action: $S = \frac{1}{2}\int d^2x d\tau \Big((\partial_\tau \chi)^2 + K(\nabla^2\chi)^2 \Big)$

Scaling operators: $\mathcal{O}_\alpha(x) = e^{i\alpha\chi(x)}$

Vacuum 2-point function

$$G_E(x_2,\tau_2;x_1,\tau_1) = \langle T_E \chi(x_2,t_2)\chi(x_1,t_1)\rangle = \int \frac{d\omega d^2p}{(2\pi)^3} \frac{e^{-i\omega(\tau_2-\tau_1)+ip\cdot(x_2-x_1)}}{\omega^2+Kp^4}$$

Equal time correlators: $\tau_2 = \tau_1 = 0$

$$G_E(x_2,x_1) = \int \frac{d\omega d^2p}{(2\pi)^3} \frac{e^{ip\cdot(x_2-x_1)}}{\omega^2+Kp^4} = \frac{1}{2\sqrt{K}} \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip\cdot(x_2-x_1)}}{p^2}$$

Equal to 2-point function of 2 dimensional free scalar $S = \sqrt{K}\int d^2x(\nabla\chi)^2$

$$\langle \mathcal{O}_{\alpha_1}(x_1)...\mathcal{O}_{\alpha_n}(x_n)\rangle = \langle e^{i\alpha_1\chi(x_1)}...e^{i\alpha_n\chi(x_n)}\rangle_{CFT}$$

Vacuum autocorrelators

Autocorrelators: $\langle \mathcal{O}_{\alpha_1}(x,t_1)\mathcal{O}_{\alpha_2}(x,t_2)\ldots\mathcal{O}_{\alpha_n}(x,t_n)\rangle$

Vacuum 2-point function:

$$G_E(\tau_2,\tau_1) = \int \frac{d\omega d^2p}{(2\pi)^3}\frac{e^{-i\omega\tau_{21}}}{\omega^2+Kp^4} \qquad \tau_{ij}=\tau_i-\tau_j$$

$$= \int \frac{d\omega dp}{(2\pi)^2}\frac{pe^{-i\omega\tau_{21}}}{\omega^2+Kp^4} \qquad d^2p = 2\pi\, p\, dp$$

$$= \frac{1}{4\sqrt{K}}\int \frac{d\omega dq}{(2\pi)^2}\frac{e^{-i\omega\tau_{21}}}{\omega^2+q^2} \qquad q=\sqrt{K}\,p^2$$

Equal to 2-point function of 2 dimensional free scalar:

$$S_{CFT} = 2\sqrt{K}\int dx d\tau \Big((\partial_\tau\chi)^2 + (\partial_x\chi)^2\Big)$$

$$G_E(\tau_2,\tau_1) = \frac{1}{4\sqrt{K}}\int \frac{d\omega dq}{(2\pi)^2}\frac{e^{-i\omega\tau_{21}}}{\omega^2+q^2+\mu^2}$$

IR regulator

$$= \frac{1}{8\pi\sqrt{K}}K_0(\mu|\tau_{21}|)$$

$$= -\frac{1}{8\pi\sqrt{K}}\Big(\log(\mu|\tau_{21}|)+\log 2-\gamma_E\Big)+O(\mu^2|\tau_{21}|^2\log(\mu|\tau_{21}|)),$$

Wick contraction gives 2 point function of scaling operators

$$\langle T_E e^{i\alpha\chi(\tau_2)}e^{i\beta\chi(\tau_1)}\rangle = e^{-\frac{1}{2}(\alpha^2 G_E(\tau_1+\epsilon,\tau_1)+\beta^2 G_E(\tau_2+\epsilon,\tau_2)+2\alpha\beta G_E(\tau_2,\tau_1))}$$

$$= (2\mu e^{-\gamma_E})^{\frac{(\alpha+\beta)^2}{16\pi\sqrt{K}}}|\tau_{21}|^{\frac{\alpha\beta}{8\pi\sqrt{K}}}\epsilon^{\frac{\alpha^2+\beta^2}{16\pi\sqrt{K}}}$$

Consider $\alpha+\beta=0$ to avoid IR divergence and absorb UV divergence into definition of scaling operator: $e_R^{i\alpha\chi} = \epsilon^{-\Delta}e^{i\alpha\chi}$

$$\Delta = \frac{\alpha^2}{16\pi\sqrt{K}}$$

Regularized 2 point function: $\langle T_E e_R^{i\alpha\chi(\tau_2)}e_R^{-i\alpha\chi(\tau_1)}\rangle = \dfrac{1}{|\tau_{21}|^{2\Delta}}$

Regularized 3 point function: $\alpha_1+\alpha_2+\alpha_3=0$

$$\langle T_E e_R^{i\alpha_1\chi(\tau_1)}e_R^{i\alpha_2\chi(\tau_2)}e_R^{i\alpha_3\chi(\tau_3)}\rangle = \epsilon^{-\Delta_1-\Delta_2-\Delta_3}e^{-\frac{1}{2}\sum_{i,j}\alpha_i\alpha_j G_E(\tau_i,\tau_j)}$$

$$= \frac{1}{|\tau_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\tau_{13}|^{\Delta_1-\Delta_2+\Delta_3}|\tau_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

Thermal state autocorrelators

Matsubara sum:

$$G_E(\tau_2,\tau_1) = T\sum_n \int \frac{d^2p}{(2\pi)^2}\frac{e^{-i\omega_n\tau_{21}}}{\omega_n^2+Kp^4}, \quad \omega_n = 2\pi nT$$

$$= \frac{1}{4\sqrt{K}}T\sum_n \int \frac{dq}{2\pi}\frac{e^{-i\omega_n\tau_{21}}}{\omega_n^2+q^2} \qquad q=\sqrt{K}\,p^2$$

Thermal 2-point function: P. Ghaemi, A. Vishwanath and T. Senthil '04

$$\langle T_E e_R^{i\alpha\chi(\tau_2)} e_R^{-i\alpha\chi(\tau_1)}\rangle = \epsilon^{-2\Delta} e^{-\frac{\alpha^2}{2}(G_E(\tau_1+\epsilon,\tau_1)+G_E(\tau_2+\epsilon,\tau_2)-2G_E(\tau_2,\tau_1))}$$

$$= \left(\frac{\pi T}{\sin(\pi T|\tau_{21}|)}\right)^{2\Delta}$$

Thermal 3-point function:

$$\langle T_E e_R^{i\alpha_1\chi(\tau_1)} e_R^{-i\alpha_2\chi(\tau_2)} e_R^{-i\alpha_3\chi(\tau_3)}\rangle$$

$$= \frac{(\pi T)^{\Delta_1+\Delta_2+\Delta_3}}{\sin(\pi T|\tau_{12}|)^{\Delta_1+\Delta_2-\Delta_3}\,\sin(\pi T|\tau_{13}|)^{\Delta_1+\Delta_3-\Delta_2}\,\sin(\pi T|\tau_{23}|)^{\Delta_2+\Delta_3-\Delta_1}}$$

Holographic correlation functions

Bulk scalar field: $$S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,\left(\partial^\mu\psi\partial_\mu\psi + m^2\psi^2\right)$$

$r \to \infty$: $$\psi(r) \to c_-\left(r^{-\Delta_-} + \ldots\right) + c_+\left(r^{-\Delta_+} + \ldots\right)$$

$$\Delta_\pm = \frac{z+2}{2} \pm \sqrt{\left(\frac{z+2}{2}\right)^2 + m^2}$$

Calculate 2-pt function of operator dual to ψ in geodesic approximation valid for large $\Delta \approx m$

$$\langle\mathcal{O}(x)\mathcal{O}(x')\rangle \approx \epsilon^{-2\Delta}e^{-\Delta\,L(x,x')}$$

UV cutoff

where *L(x,x')* is the length of the shortest geodesic connecting *x* and *x'*

$$L(x,x') = \int_{\lambda_1}^{\lambda_2} d\lambda\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = \lambda_2 - \lambda_1$$

Geodesics in Lifshitz spacetime

Lifshitz metric: $$ds^2 = \frac{d\tau^2}{u^4} + \frac{du^2 + d\mathbf{x}^2}{u^2}$$

Geodesic equations:

$$\frac{1}{u^4}\dot{\tau} = E,$$

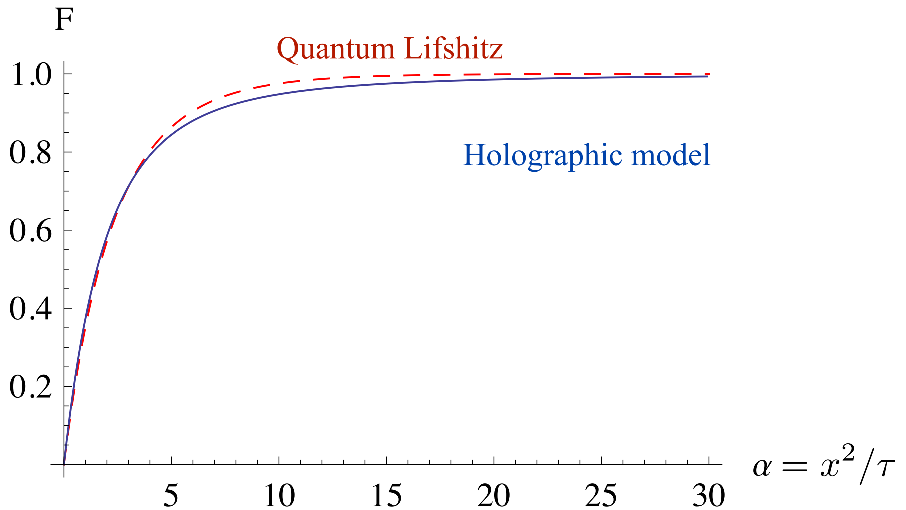
$$\frac{1}{u^2}\dot{x} = p,$$ (assume endpoints lie on *x* axis)

$$\frac{\dot{u}^2}{u^2} = 1 - u^4 E^2 - u^2 p^2$$

The resulting 2- point function has a scaling form:

$$G_E(r,\tau) \propto \frac{1}{x^{4\tilde{\Delta}}} F(x^2/\tau)^{2\tilde{\Delta}} \qquad \tilde{\Delta} = m/2$$

$$G_E(x,t) \propto \frac{1}{\tau^{2\tilde{\Delta}}}\left(1 - \frac{2\tilde{\Delta}}{\pi}\frac{x^2}{\tau} + \ldots\right) \qquad x^2 \ll \tau$$



Three-point function in Lifshitz spacetime

Consider a bulk theory with a 3-point vertex: $-\frac{\lambda}{3!}\int d^d x\sqrt{g}\phi_1(x)\phi_2(x)\phi_3(x)$

To leading order the 3 point function is then given by:

$$G_3(\tau_1,\tau_2,\tau_3) = -\lambda\int d^d x\sqrt{g}G_{BB}(\tau_1,0;z,\tau,\mathbf{x})G_{BB}(\tau_2;0;z,\tau,\mathbf{x})G_{BB}(\tau_3,0;z,\tau,\mathbf{x})$$

Bulk-to-boundary propagator: $G_{BB}(\tau_j,0;z,\tau,x) \propto e^{-m_j L(\epsilon,\tau_j,0;z,\tau,\mathbf{x})}$

The Lifshitz metric can be rewritten: $ds^2 = \frac{1}{z^2}\frac{1}{y^2}(d\tau^2+dy^2)+\frac{1}{(zy)^{2/z}}d\mathbf{x}^2$

Saddle point evaluation: $G_3(\tau_1,\tau_2,\tau_3) = \frac{C(\Delta_1,\Delta_2,\Delta_3)}{\tau_{12}^{\Delta_1+\Delta_2-\Delta_3}\tau_{13}^{\Delta_1+\Delta_3-\Delta_2}\tau_{23}^{\Delta_2+\Delta_3-\Delta_1}}$

$$C(\Delta_1,\Delta_2,\Delta_3) = -\lambda\gamma^{\frac{\Delta_1+\Delta_2+\Delta_3}{2}}2^{-\Delta_1-\Delta_2-\Delta_3}\Delta_1^{-\Delta_1}\Delta_2^{-\Delta_2}\Delta_3^{-\Delta_3}(\Delta_2+\Delta_3-\Delta_1)^{-\Delta_1}\times$$
$$\times(\Delta_1+\Delta_3-\Delta_2)^{-\Delta_2}(\Delta_1+\Delta_2-\Delta_3)^{-\Delta_3}$$

This has the same form as a 3 point function in a 1+1 dimensional CFT!

Thermal autocorrelators $z = 2,\ d = 2$

Lifshitz black hole: $$ds^2 = \frac{f(u)}{u^4}d\tau^2 + \frac{du^2}{u^2 f(u)} + \frac{d\mathbf{x}^2}{u^2}, \quad f(u) = 1 - u^4$$

Thermal 2-point function: $$G_E(\tau_2, x; \tau_1, x) \propto \left(\frac{\pi T}{\sin(\pi T|\tau_{21}|)}\right)^{2\tilde{\Delta}}$$

Identical to thermal autocorrelator in the quantum Lifshitz model!

Geodesics that contribute to autocorrelator only involve $g_{\tau\tau},\ g_{rr}$

Mapping to BTZ black hole: $$ds^2_{2,Lif} = \left(1 - \frac{u^4}{u_H^4}\right)\frac{d\tau^2}{u^4} + \frac{du^2}{u^2(1 - \frac{u^4}{u_H^4})}$$

$$ds^2_{2,BTZ} = \frac{1}{y^2}\left[\left(1 - \frac{y^2}{y_H^2}\right)d\tau^2 + \frac{dy^2}{1 - \frac{y^2}{y_H}^2}\right]$$

$$y = \frac{u^2}{2}, \quad ds^2_{2,Lif} = \frac{1}{4}ds^2_{2,BTZ} \qquad y_H = u_H^2/2$$

BTZ geometry is dual to a thermal state in 1+1 dimensional CFT

Dynamical solutions   EMD model, $d = 2$

Keränen, Keski-Vakkuri, & L.T. '11

Rewrite static black brane solution using $dv = dt + \frac{r^{-z-1}}{b(r)} dr$

$$ds^2 = -r^{2z} b(r) dv^2 + 2r^{z-1} dv\, dr + r^2 d\mathbf{x}^2$$

$$b(r) = 1 - \tilde{m}\left(\frac{r_0}{r}\right)^{z+2} + \frac{\tilde{\rho}^2}{4z}\left(\frac{r_0}{r}\right)^{2z+2} \qquad F^{(2)}_{rt} = z\tilde{\rho}\, r_0^{z-1}\left(\frac{r_0}{r}\right)^{z+1}$$

$$e^{\phi} = \left(\frac{r}{r_0}\right)^{2\sqrt{z-1}} \qquad F^{(1)}_{rt} = 2r_0^{z-1}\sqrt{(z+2)(z-1)}\left(\frac{r}{r_0}\right)^{z+1}$$

Lifshitz-Vaidya solution: $\tilde{m} \to \tilde{m}(v)\,, \quad \tilde{\rho} \to \tilde{\rho}(v)$

$\tilde{m}(v)\,,\ \tilde{\rho}(v)$ determined by incoming energy and charge density

The time and radial parts of metric can be mapped to a BTZ-Vaidya solution

2-point autocorrelators can then be obtained from autocorrelators calculated in BTZ-Vaidya spacetime

Balasubramanian, Bernamonti, Craps, Keränen, Keski-Vakkuri, Müller, L.T., Vanhoof '12

Summary

- Black branes in asymptotically Lifshitz spacetime provide a window onto finite temperature effects in strongly coupled models with anisotropic scaling.
- Analytic black brane solutions are available for all physical values of d and z in an Einstein-Maxwell-Dilaton (EMD) model, at the price of having a strongly coupled background gauge field (that does not couple directly to other gauge fields or matter).
- Autocorrelators of scaling operators are identical in the quantum Lifshitz model and in a strongly coupled Lifshitz model with a holographic dual.
- Vacuum autocorrelators in the holographic model at any value of z and d can be expressed in terms of autocorrelators of a 1+1 dimensional CFT.
- Thermal autocorrelators at $z = d$ can also be expressed in terms of the 1+1 dimensional CFT.
- These methods can be used to study out-of-equilibrium phenomena in Lifshitz models:
  - mass quench in the quantum Lifshitz model
  - holographic quench in Lifshitz-Vaidya spacetime