Seeing Asymptotic Freedom in an Exact Correlator of a Large-N Matrix Field Theory

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Principal Chiral Sigma Model (PCSM) in 1+1 dimensions:

Lagrangian: $\mathscr{L} = \frac{N}{2g_0^2}\mathrm{Tr}\,\partial_\mu U^\dagger \partial^\mu U, \quad U \in \mathrm{SU}(N).$

Currents (no central charge):

$$j_\mu^{\mathrm{Left}}(x)_b = \mathrm{i}\mathrm{Tr}\, t_b\, \partial_\mu U(x)\, U(x)^\dagger \;, \quad j_\mu^{\mathrm{Right}}(x)_b = \mathrm{i}\mathrm{Tr}\, t_b\, U(x)^\dagger \partial_\mu U(x) \;.$$

Hamiltonian:

$$H = \int dx^1 \frac{1}{2g_0^2}\{[j_0^{\mathrm{L}}(x)_b]^2 + [j_1^{\mathrm{L}}(x)_b]^2\} = \int dx^1 \frac{1}{2g_0^2}\{[j_0^{\mathrm{R}}(x)_b]^2 + [j_1^{\mathrm{R}}(x)_b]^2\}.$$

Why study large-N PCSM?

Answer 1. Interesting in its own right!

- Asympt. free, matrix QFT. *NO* conf. inv. or supersymmetry. $1/N \approx 0$ sad.-pt. methods fail. Unit S matrix, but not free QFT.
- 2-point function valid at all length scales. It thereby provides a yardstick that other methods can be compared to (should they succeed).
- Temperature> 0 (A. Cortés Cubero, PR**D91** 105025 (2015)).

Answer 2. Applications! A. Cortés Cubero and me.

- Yang-Mills string tensions and mass gaps at *WEAK COUPLING* in d=2+1. Not fully Lorentz inv., but no worse than Hamiltonian strong-coupling exp. in lattice gauge theory.
- Massive Yang-Mills in d=1+1. Dynamical mass reduction (the actual masses are corrections to twice the PCSM mass - *not* the YM mass).

A quote concerning SU(∞) PCSM:

"A quantitative check of these guesses has not yet been done. There are no doubts, however, that the mystery of the large N limit for chiral fields will soon be resolved." - A. M. Polyakov, **Gauge Fields and Strings**, Harwood Academic Publishers (1987).

Scaling field is $\Phi(x) \sim Z^{-1/2}\, U(x)$ (equality only in Green's functions):

$$\langle 0|\, \Phi(0)_{ba}\, |A, \theta, d, c\rangle_{\mathrm{in}} = \frac{1}{\sqrt{N}} \delta_{ac}\delta_{bd},$$

where A indicates an antiparticle and θ is its rapidity.

The fields Φ, $\Phi^{\dagger}$ are *not* unitary, in general ($N \times N$ complex matrices).

Pert. RG: $U = e^{iA^\alpha t_\alpha}$, $L = \frac{N}{2g_0^2}\, g_{\alpha\beta}(A)\, \partial_\mu A^\alpha \partial^\mu A^\beta$,

$$g_{\alpha\beta}(A) = \left[\tfrac{\cosh(A\cdot f - \mathbb{1})}{(A\cdot f)^2}\right]_{\alpha\beta} = \delta_{\alpha\beta} + \tfrac{1}{12} f_{\alpha\rho\gamma} f_{\gamma\sigma\beta} A^\rho A^\sigma + \cdots$$

$$= \delta_{\alpha\beta} + \frac{1}{3} R_{\rho\alpha\beta\sigma} A^\rho A^\sigma + \cdots \text{ (Riemann normal coordinates).}$$

The (time-ordered) correlator in perturbation theory:

$$\frac{1}{N}\mathrm{Tr}\langle U(x)U(0)^\dagger\rangle \simeq \exp -\frac{1}{2N}\mathrm{Tr}\langle A(x)A(0)\rangle = \exp\left[\frac{N^2-1}{4\pi N}\frac{g_0^2}{N}\ln(|x|\Lambda)\right].$$

Beta function: $\frac{\partial g_0^2}{\partial \ln\Lambda} = -\beta_1 g_0^4 + \cdots, \;\; \beta_1 = \frac{1}{8\pi}$.

Anomalous dimension: $\frac{\partial \ln G(|x|,\Lambda)}{\partial \ln\Lambda} = \gamma_1 g_0^2 + \cdots, \;\; \gamma_1 = \frac{N^2-1}{4\pi N^2}$.

Universal behavior at short distances. As $\Lambda \to \infty$ (criticality),

$$G(|x|,\Lambda) \simeq G(\Lambda|x|) \simeq C \ln^{\gamma_1/\beta_1}(\Lambda|x|).$$

But $\gamma_1/\beta_1 = 2\frac{N^2-1}{N^2} \to 2$, as $N \to \infty$.

The 2-point function, and more results (not discussed here), were obtained combining the $1/N$-expansion with the form-factor bootstrap. Integrability of the PCSM is used, but perhaps is inessential.

It is striking how the short-distance behavior $G(m|x|) \to C \ln^2(m|x|)$, emerges from the expression for the Wightman function, with *no use of perturbation theory.*

The S matrix bootstrap in 1+1 dimensions.

In integrable QFT's, unitarity and factorization (Yang-Baxter equation) determine the exact S matrix, *up to CDD ambiguities*. Often (not always) these ambiguities can be eliminated by:

- Kinematic restrictions, e.g., the sine-law for bound states.
- Other knowledge about the spectrum.
- Comparison with perturbation theory in the coupling constant or (in the case of isovector models) 1/n, n=no. of. components.

Alternatively, use Bethe's *Ansatz*, which works for certain systems, e.g., spin chains. Often tricky to identify the Bethe-*Ansatz*-solvable model with a QFT.

The simplest two-particle S matrix is that of the Ising field theory, $L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda(\phi^2 - \beta)$, $\lambda \to \infty$. It is $S = -1$.

The form-factor bootstrap in 1+1 dimensions.

Early determinations of some form factors for sinh-Gordon, sine-Gordon, Ising, O(n) sigma/Gross-Neveu,... made from PCT (Watson's theorem), the reduction formula, Lorentz invariance and crossing. Later, F. Smirnov formulated axioms.

Again, there are ambiguities. Sometimes we can fix these by comparison with perturbation theory, 1/n-expansions, etc. Models with rich spectra of bound states, sine-Gordon, SU(N) chiral Gross-Neveu, PCSM, are technically harder than others.

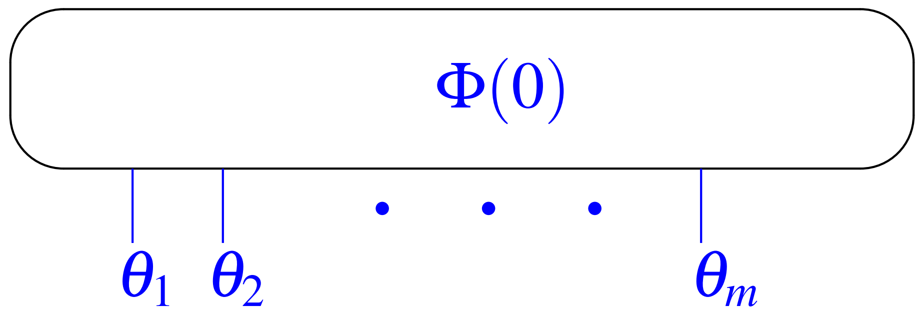
$$FF = \langle 0|\Phi(x)|p_1,\ldots,p_m\rangle_{\rm in} = \langle 0|\Phi(x)|\theta_1,\ldots,\theta_m\rangle_{\rm in},$$

$$\langle 0|\Phi(x)\Phi(0)|0\rangle = \sum_X \langle 0|\Phi(x)|X\rangle_{\rm in}\ {}_{\rm in}\langle X|\Phi(0)|0\rangle.$$

To find correlators, *all* form factors are needed. 1st few terms agrees with Monte-Carlo results.

$$\langle 0 \,|\, \Phi(0) \,|\, \theta_1, \theta_2, \ldots, \theta_m \rangle_{\rm in}$$

$=$



For Ising FT, in the unmagnetized phase, $\Phi = Z^{-1/2}\phi$, $\langle 0|\Phi|\theta\rangle = 1$ (normalization) and

$$\langle 0|\Phi(x)|\theta_1, \ldots, \theta_m\rangle_{\rm in} \sim \prod_{j<k} \tanh\frac{\theta_j - \theta_k}{2} \,.$$

The first few form factors work fairly close to the critical point, giving approximately $\langle \Phi(x)\Phi(0)\rangle \sim |x|^{-1/4}$ (Yurov and Zamolodchikov, Cardy and Mussardo 1990).

S Matrix of PCSM

Polyakov+Wiegmann (1983), Abdalla^2+Lima Santos (1984), Wiegmann (1984)

Spectrum: $m_r = m\frac{\sin(\pi r/N)}{\sin(\pi/N)}$, $r = 1,\dots,N-1$.

Elementary color dipoles $r=1$ ($q\bar{q}$), bound states $r>1$. Elementary antiparticle: $r=N-1$. $\theta=\theta_{12}=\theta_1-\theta_2$, $m\cosh\theta_j=E_j$, $m\sinh\theta_j=p_j$. For $r\ll N$, $m_r=m_{N-r}\approx mr$, and binding energy is zero. Surviving bound states are heavy, $m_r\sim N$, except $m_{r-1}=m_1=m$.

(r=1) by (r=1) S-matrix, sans kinematic factors:

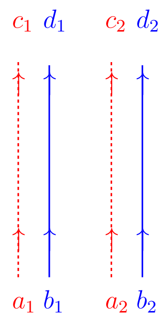
$$S_{11}(\theta)=\frac{\sin(\theta/2-\pi\mathrm{i}/N)}{\sin(\theta/2+\pi\mathrm{i}/N)}\,S_{\mathrm{CGN}}(\theta)\otimes S_{\mathrm{CGN}}(\theta),$$

$$S_{\mathrm{CGN}}(\theta)=\frac{\Gamma(\mathrm{i}\theta/2\pi+1)\Gamma(-\mathrm{i}\theta/2\pi-1/N)}{\Gamma(\mathrm{i}\theta/2\pi+1-1/N)\Gamma(-\mathrm{i}\theta/2\pi)}\left(\mathbb{1}-\frac{2\pi\mathrm{i}}{N\theta}P\right).$$

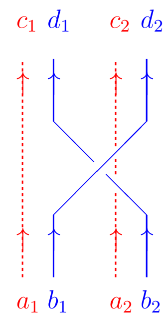
Crossing $\theta\to\pi\mathrm{i}-\theta$ and fusion, give full S matrix. **The residue of bound-state poles is order $1/N$.**

$$S_{PP}(\theta)^{a_1b_1;a_2b_2}_{c_2d_2;c_1d_1} = \left[1+O\left(\frac{1}{N^2}\right)\right]$$

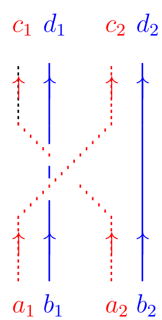
$\times \Bigg[$



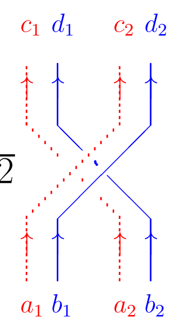
$-\dfrac{2\pi\mathrm{i}}{N\theta}$



$-\dfrac{2\pi\mathrm{i}}{N\theta}$



$-\dfrac{4\pi^2}{N^2\theta^2}$



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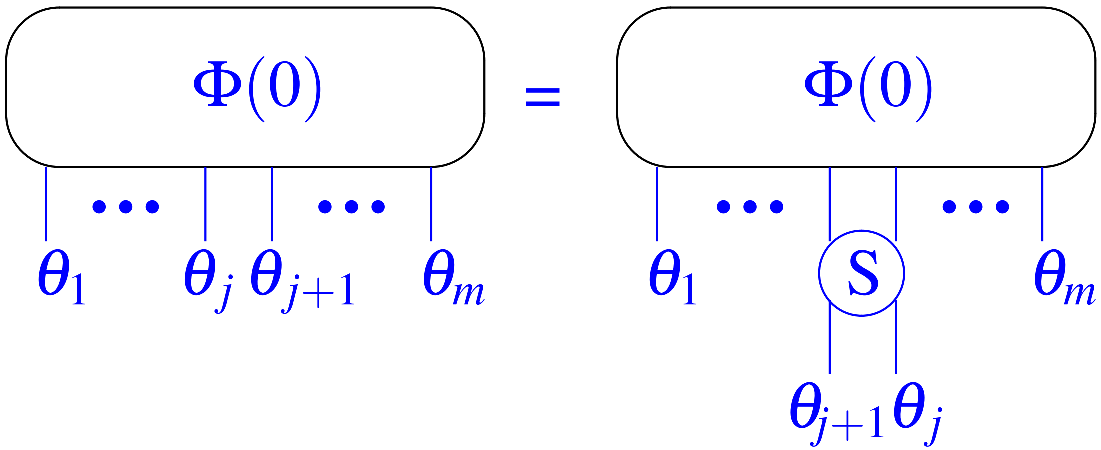
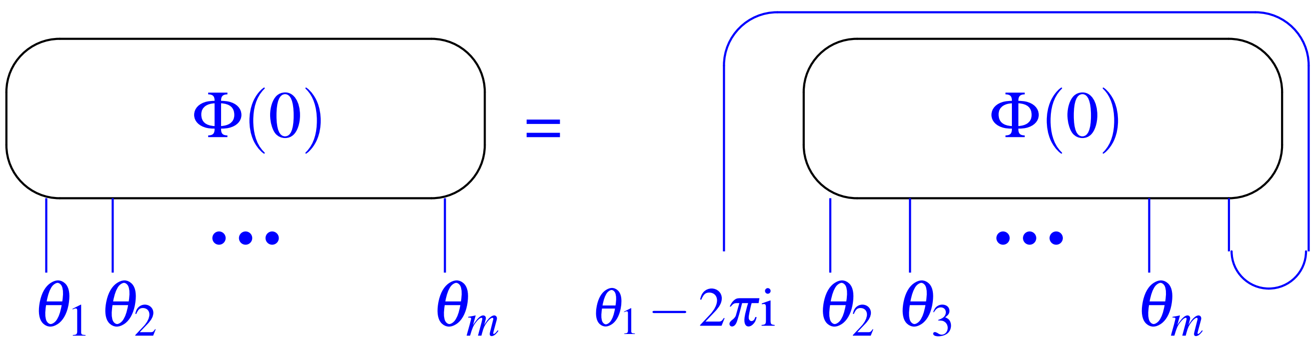
Smirnov's axioms.

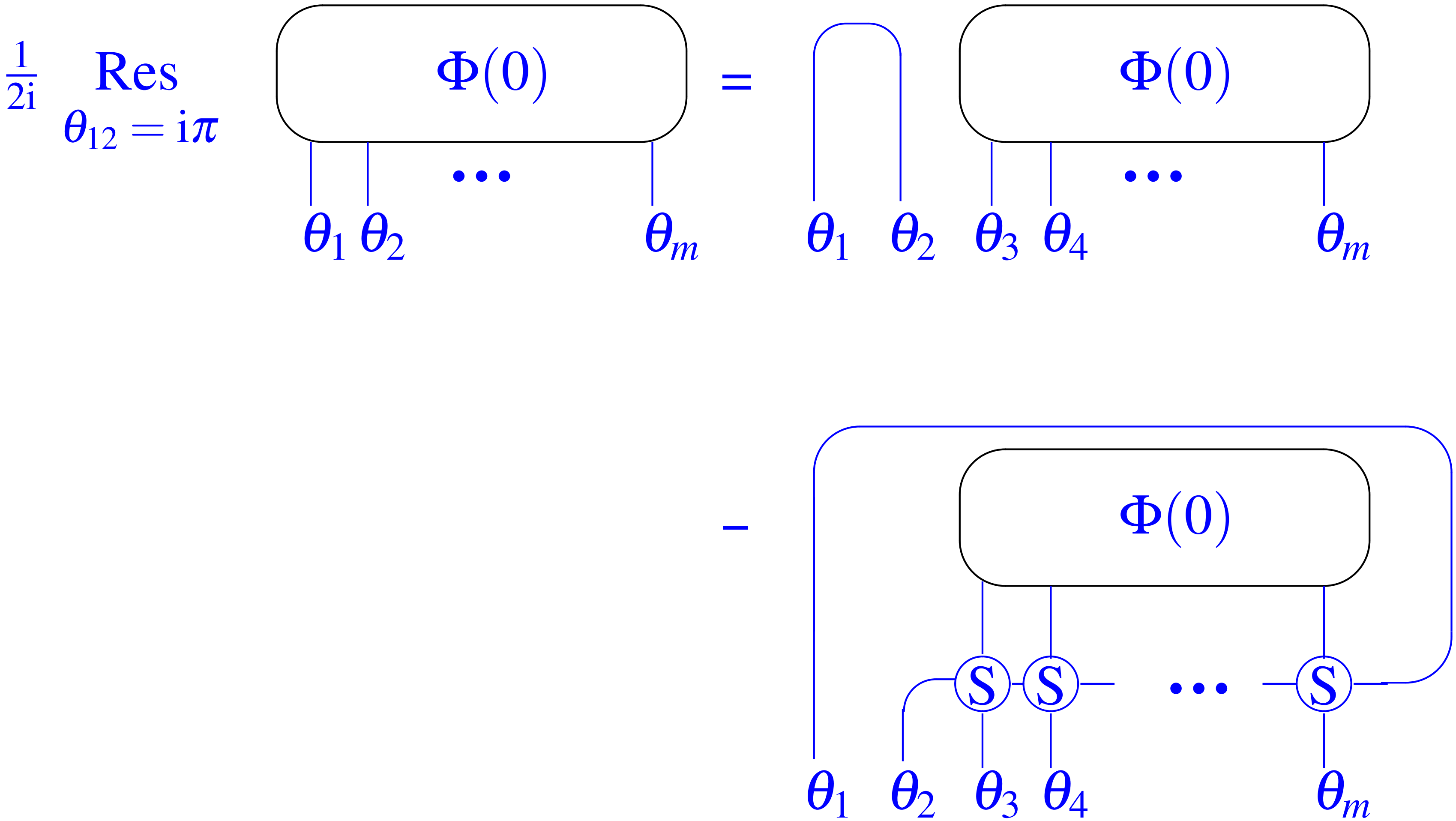
1. ***Scattering Axiom*** **(Watson's theorem, implied by PCT):**



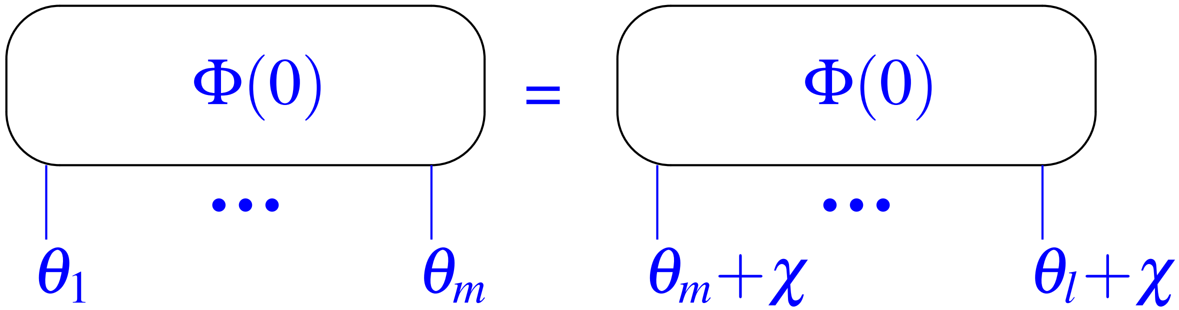
2. ***Periodicity Axiom*** **(generalized crossing):**



3. ***Annihilation Pole Axiom*** **(reduction formula):**



4. ***Lorentz Invariance Axiom*** **(for boost χ):**



5. ***Bound-State Pole Axiom*****:**

There are poles on the imaginary axis of rapidity differences θ_{jk}, due to bound states.

6. ***Maximal Analyticity Axiom*****:**

Form factors are holomorphic, except possibly for bound-state poles or annihilation poles, for rapidity differences θ_{jk}, in the complex strip $0 < \Im m\ \theta_{jk} < 2\pi$. More guideline than axiom (may fail!).

Comments: Axiom 1. is nontrivial because contractions from S_{PA} are order N. Axiom 2. is valid, despite the breakdown of crossing.

Why isn't the large-N limit trivial?

- The S matrix becomes unity in this limit. All the interactions are powers of $1/N$.
- But acting on a form factor with S_{PA} (but not S_{PP} or S_{AA}) produces new factors of N. An effective two-particle S matrix appears, which is a pure phase (which depends upon rapidity difference).
- After appropriate color contractions, an excitation *can* scatter at most two excitations as $N \to \infty$. Each particle (antiparticle) can scatter one or two antiparticles (particles). We'll list Smirnov's axioms in a moment, where this result is the application of Axiom 1. (Watson's theorem).
- In isovector models, an excitation can scatter at most one other excitation. Axioms 1. (Watson's theorem) and 2. (periodicity, from crossing) imply triviality.

The master field.

Although Φ is not a free field, there is an associated free field!

Zamolodchikov Algebra of Generalized Creation Operators

$$\mathfrak{A}^{\dagger}_{P}(\theta_1)_{a_1b_1}\,\mathfrak{A}^{\dagger}_{P}(\theta_2)_{a_2b_2} \;=\; S_{PP}(\theta_{12})^{c_2d_2;c_1d_1}_{a_1b_1;a_2b_2}\,\mathfrak{A}^{\dagger}_{P}(\theta_2)_{c_2d_2}\,\mathfrak{A}^{\dagger}_{P}(\theta_1)_{c_1d_1}$$

$$\mathfrak{A}^{\dagger}_{A}(\theta_1)_{b_1a_1}\,\mathfrak{A}^{\dagger}_{A}(\theta_2)_{b_2a_2} \;=\; S_{AA}(\theta_{12})^{d_2c_2;d_1c_1}_{b_1a_1;b_2a_2}\,\mathfrak{A}^{\dagger}_{A}(\theta_2)_{d_2c_2}\,\mathfrak{A}^{\dagger}_{A}(\theta_1)_{d_1c_1}$$

$$\mathfrak{A}^{\dagger}_{P}(\theta_1)_{a_1b_1}\,\mathfrak{A}^{\dagger}_{A}(\theta_2)_{b_2a_2} \;=\; S_{PA}(\theta_{12})^{d_2c_2;c_1d_1}_{a_1b_1;b_2a_2}\,\mathfrak{A}^{\dagger}_{A}(\theta_2)_{d_2c_2}\,\mathfrak{A}^{\dagger}_{P}(\theta_1)_{c_1d_2}\,,$$

$$P = \text{Particle}\;, \quad A = \text{Antiparticle.}$$

Associativity of the Zamolodchikov algebra implies the Yang-Baxter equation.

As $N \to \infty$, $\mathfrak{A}^{\dagger}$'s commute. The master field is the free field:

$$M(x) = \int \frac{d\theta}{4\pi} \left[\mathfrak{A}_P^{\dagger}(\theta) e^{\mathrm{i} p\cdot x} + \mathfrak{A}_A(\theta) e^{-\mathrm{i} p\cdot x} \right] .$$

The form factors yield a functional Taylor series of $\Phi(x)$ in terms of $M(x)$.

In principle, we can find the Hamiltonian and the Schrödinger vacuum functional:

- Invert the functional Taylor series to find $\mathfrak{A}_A(\theta)$, $\mathfrak{A}_P(\theta)$ in terms of $\Phi(x)$.
- The Hamiltonian is now the sum of s.h.o. Hamiltonians.
- Find $\mathfrak{A}_A(\theta)$, $\mathfrak{A}_P(\theta)$ in the Schrödinger representation.
- The vacuum satisfies two first-order functional equations, namely that it is annihilated by $\mathfrak{A}_A(\theta)$, $\mathfrak{A}_P(\theta)$.

Form factors of the SU(∞) PCSM.

PR**D84** (2011) 105005, PR**D86** (2012) 045023.

$$\langle 0|\Phi(0)_{b_0 a_0}\mathfrak{A}_A(\theta_1)_{b_1 a_1}\cdots\mathfrak{A}_A(\theta_{M-1})_{b_{M-1}a_{M-1}}\mathfrak{A}_P(\theta_M)_{a_M b_M}\cdots\mathfrak{A}_P(\theta_{2M-1})_{a_{2M-1}b_{2M-1}}|0\rangle$$

$$=\frac{1}{N^{M-1/2}}\sum_{\sigma,\tau\in S_M}F_{\sigma\tau}(\theta_1,\theta_2,\ldots,\theta_{2M-1})\prod_{j=0}^{M-1}\delta_{a_j\,a_{\sigma(j)+M}}\delta_{b_j\,b_{\tau(j)+M}},$$

where $F=F^0+O(1/N)$,

$$F^0_{\sigma\tau}(\theta_1,\theta_2,\ldots,\theta_{2M-1})=\frac{(-4\pi)^{M-1}K_{\sigma\tau}}{\prod_{j=1}^{M-1}[\theta_j-\theta_{\sigma(j)+M}+\pi\mathrm{i}][\theta_j-\theta_{\tau(j)+M}+\pi\mathrm{i}]},$$

and where $K_{\sigma\tau}=1$, if $\sigma(j)\neq\tau(j)$, for all j and $K_{\sigma\tau}=0$, otherwise.

This is the *minimal* choice of FF for SU(N)×SU(N) symmetry.

Putting everything together:

Combine these form factors and completeness:

$$\frac{1}{N}\langle 0|\Phi(x)\Phi(0)^{\dagger}|0\rangle = \frac{1}{N}\sum_{X}\langle 0|\Phi(x)|X\rangle_{\rm in}\ {}_{\rm in}\langle X|\Phi(0)^{\dagger}|0\rangle \implies$$

$$\tfrac{1}{N}\langle 0|\mathrm{Tr}\,\Phi(x)\Phi(0)^{\dagger}|0\rangle = Z^{-1}\frac{1}{N}\langle 0|\mathrm{Tr}\,U(x)U(0)^{\dagger}|0\rangle$$

$$= \int \frac{d\theta_1}{4\pi}\,e^{ix\cdot p_1} + \frac{1}{4\pi}\sum_{l=1}^{\infty}\int d\theta_1\cdots d\theta_{2l+1}\,e^{\mathrm{i}x\cdot\sum_{j=1}^{2l+1}p_j}\prod_{j=1}^{2l}\frac{1}{(\theta_j-\theta_{j+1})^2+\pi^2}$$

$+\ \ O(N^{-1})$. ET VOILÀ!

Digust, existential nausea, self-loathing: How do we know *minimal* FF are the PCSM's? Convergence? Short distances? **We must test this expression.**

Eucl. t-ordered correlators, PR**D90**, 125038 (2014),

Set $x^1 = 0$, $x^0 = \mathrm{i}R$, $R > 0$. Then $e^{\mathrm{i}p_j \cdot x} = e^{-mR\cosh\theta_j}$. For $mR \ll 1$,

$$e^{-mR\cosh\theta_j} \approx \begin{cases} 1, & -L < \theta_j < L \\ 0, & \text{otherwise} \end{cases}, \quad L = \ln\frac{1}{mR}$$

Walls in rapidity space at $\theta_j = \pm L$ contain the Feynman-Wilson gas in the Ising model (Cardy and Mussardo, Yurov and Zamolodchikov 1990). PCSM is a Feynman-Wilson *polymer*.

$L \gg 1$ requires Lévy's central-limit theorem (thanks to Timothy Budd!). Introduce $u_j = \theta_j / L$.

$$G(mR) = \frac{L}{2\pi}$$
$$+\frac{L}{4\pi}\sum_{l=1}^{\infty}\int_{-1}^{1} du_1 \cdots \int_{-1}^{1} du_{2l+1} \prod_{j=1}^{2l} \frac{1}{L[(u_j - u_{j+1})^2 + (\pi/L)^2]}.$$

Extend $u, u' \in (-\infty, \infty)$. Operators $P(a)$, defined by $\langle u'|P(a)|u\rangle = (a/\pi)[(u'-u)^2 + a^2]^{-1}$. Poisson semigroup: $P(a)P(b) = P(a+b)$. Specifically, $P(a) = \exp -a\Delta^{1/2}$, where $\Delta^{1/2} = \sqrt{-d^2/du^2}$.

Define

$$\langle u'|\exp -\frac{\pi}{L}H(L)|u\rangle = \frac{1}{L[(u_j - u_{j+1})^2 + (\pi/L)^2]}$$

on the interval $u \in (-1, 1)$. Then $H(L) = \Delta^{1/2} + O(1/L)$, where now $\Delta^{1/2}$ is the self-adjoint extension.

Short-distance (large-L) behavior.

Discrete spectrum: $\Delta^{1/2}\phi_n(u) = \lambda_n \phi_n(u)$. For large L:

$$G(mR) = \frac{L}{\pi}\int_{-1}^{1} du' \int_{-1}^{1} du \, \langle u'| \left[1 - e^{-2\pi H(L)/L}\right]^{-1} |u\rangle$$
$$= C_2 L^2 + C_1 L + C_0 + C_{-1} L^{-1} + \cdots ,$$

where

$$C_2 = \frac{1}{8\pi^2} \sum_n \left| \int_{-1}^{1} du \, \phi_n(u) \right|^2 \lambda_n^{-1}.$$

E. Katzav pointed out to me that C_2 is the integral over $(-1,1)$ of the *mean first passage time* of a Lévy flight, which implies $C_2 = 1/(16\pi)$ (not yet published).

The correlator obtained from the minimal form factors is *finite* and has the correct large-distance and short-distance behavior. Perturbation theory was not used to find this result.

What's Next?

1. Much more to do for PCSM.

- Introduce an external field to find the effective action at different length scales, from our axiomatic approach. This would prove that the FF *are* those of PCSM, standing the Clay-type problem on its head (person who solves it for QCD loses $\$10^6$). Should also yield two-loop β-function.
- Temperature > 0. A. Cortés Cubero has studied FF for $T_{\mu\nu}$ and correlators of Φ.
- The $1/N$-expansion is not known beyond the leading order.
- There are integrable supersymmetric generalizations. One way to include Bose-Fermi interactions.
- Finite volume with twisted boundary conditions and large-N reduction.

What's Next?

2. Application to gauge theories with $d = 2+1, 1+1$.

- How I entered this game: used FF of SU(2) PCSM current to study SU(2) YM_{2+1} at weak coupling. Coupling is weak, but second coupling (coefficient of one term in H) is *extra weak*. 2 couplings helps to find $q\bar{q}$-potential, spectrum. A. Cortés Cubero used SU(N) current FF (which he found earlier) to do SU(N) .
- Possible to beat the crossover from $1+1$-dimensional behavior to $2+1$ dimensional behavior, if certain correlators can be found.
- Axel and I found low-lying spectum of massive YM_{1+1}, semi-classically. Should be redone with the Bethe-Salpeter eq. Some temp.> 0 work has been done by Axel.
- No dimensional crossover, if $d = (1+\varepsilon)+1$. Earlier attempt was inconclusive (Dine, Litwin, McLerran 1981).

What's Next?

3. Bootstrap theories without integrability, $d > 1 + 1$.

- Large-N expansion of unitary S matrices. Use two-particle intermediate state approx. Impose $s \leftrightarrow t$ crossing. CDD ambiguities remain, but that is a good thing (classification). Scalar-scalar (π-π) scattering (Chew, Mandelstam 1960) fails as phenomenology, but may succeed as field theory.
- Form factors are severely constrained by the form of the S matrix. LSZ reduction formula, dispersion relations (Bogoliubov, Bremmerman, Ohme, Taylor, Jost, Lehman, Dyson 1950's).
- In massive theories, form factors (even if we can't find them all) yield rapidly-convergent expressions for correlators.

THANK YOU!