Holographic One-point Functions

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Based on:

- C. K., G. Semenoff, and D. Young, arXiv: 1210.7015, JHEP 1301 (2013) 117
- M. de Leeuw, C. K., and K. Zarembo , arXiv:1506.06958, JHEP 1508 (2015) 098
- I. Buhl-Mortensen, M. de Leeuw, C.K., and K. Zarembo, work in progress

Workshop on Holography and Condensed Matter

Perugia, September 24th , 2015

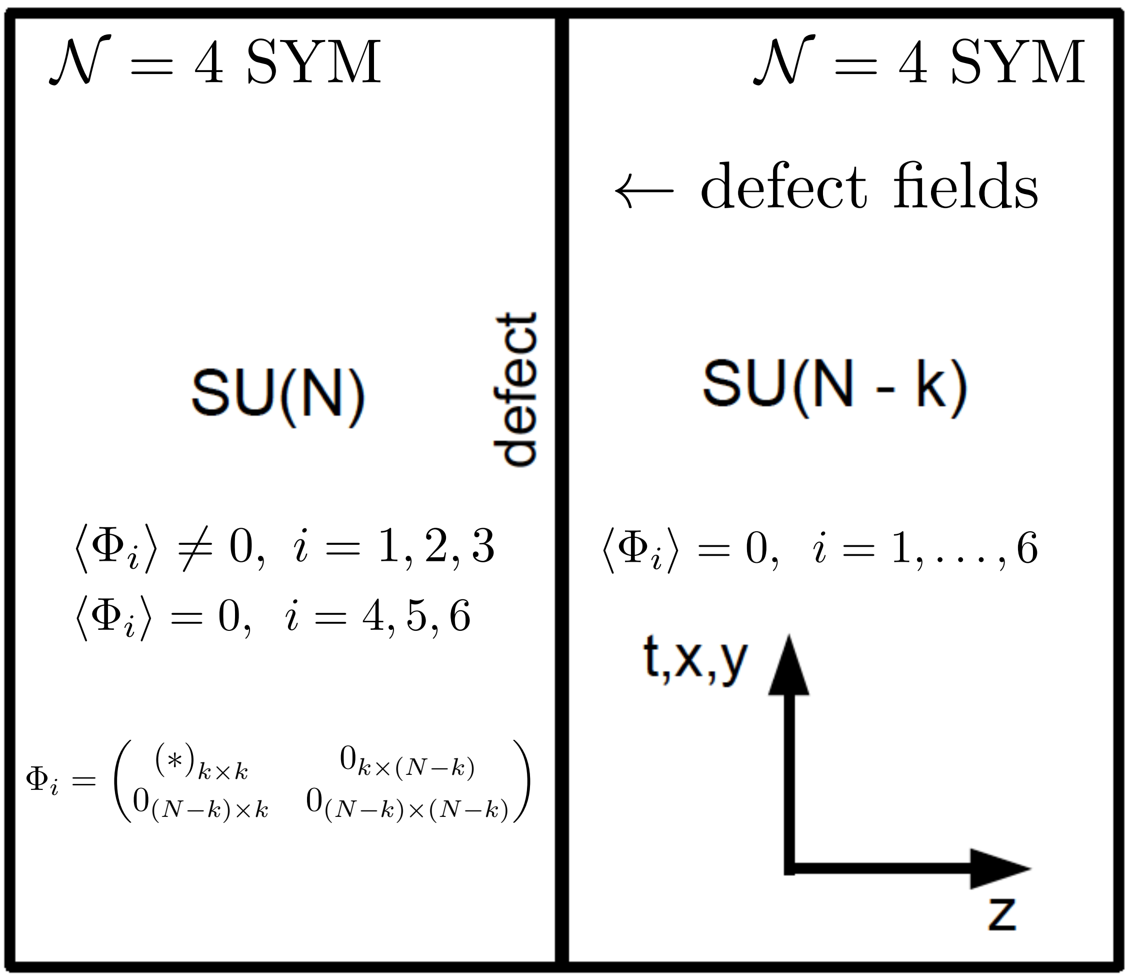
Relation to condensed matter

- Holographic set-up of relevance for CMT (probe-brane/dCFT)
- Integrable spin chain at the heart of the problem
- Use of Matrix Product States gives convenient formulation of the problem
- Néel state plays a prominent role

Plan of the talk

- The holographic set-up
- One-point functions (mainly field theory side)
- Fundamentals of the integrable Heisenberg spin chain
- Calculating one-point functions of the dCFT using integrability
- Open problems/Conclusion

AdS/dCFT --- The field theory side

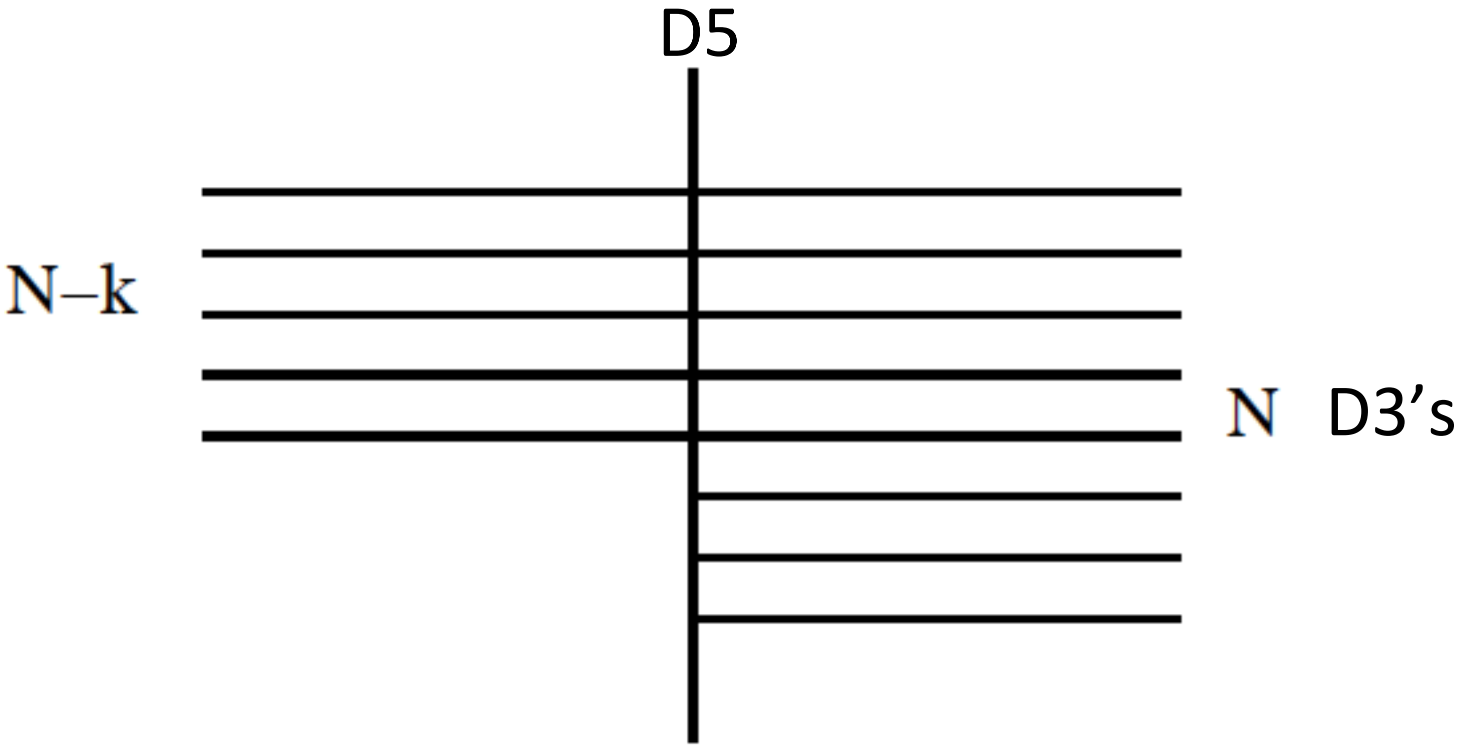


deWolfe, Freedman & Ooguri '01

Our interest: Tree-level one-point functions of single trace operators built from bulk scalar fields

AdS/dCFT --- The string theory side

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D3$	$\times$	$\times$	$\times$	$\times$						
$D5$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$			



Geometry of D5 brane: $AdS_4 \times S^2$

Background gauge field: k units of magnetic flux on S^2 Karch & Randall '01

Vev's and tree level one-point functions

$$\Phi_i^{\rm cl} \neq 0, \quad i = 1,2,3, \qquad \Phi_4^{\rm cl} = \Phi_5^{\rm cl} = \Phi_6^{\rm cl} = 0$$

$$A_\mu = 0, \qquad \Psi_A = 0$$

Classical e.o.m.:

(z is distance to defect)

$$\frac{d^2\Phi_i^{\rm cl}}{dz^2} = \left[\Phi_j^{\rm cl}, \left[\Phi_j^{\rm cl}, \Phi_i^{\rm cl}\right]\right].$$

Constable, Myers & Tafjord '99

Solution: $$\Phi_i^{\rm cl} = \frac{1}{z}\begin{pmatrix} (t_i)_{k\times k} & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{pmatrix}, \quad i = 1,2,3$$

where t_i constitute a k-dimensional irreducible repr. of SU(2) and where z<0. (Nahm eqns. also fulfilled.)

Op's with tree-level 1-point functions built from $\Phi_i, i = 1,2,3$

Tree level one-point functions

$$\Phi_i^{\text{cl}} \neq 0, \quad i = 1,2,3, \qquad \Phi_4^{\text{cl}} = \Phi_5^{\text{cl}} = \Phi_6^{\text{cl}} = 0$$

$$\Phi_i^{\text{cl}} = \frac{1}{z} \begin{pmatrix} (t_i)_{k\times k} & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{pmatrix}, \; i = 1,2,3$$

Consider SU(2) subsector: $Z = \Phi_1 + i\Phi_4, \; W = \Phi_2 + i\Phi_5$

$$\begin{aligned} \langle O \rangle &= \langle Tr(ZZZWWZZ...W) \\ &= \frac{1}{z^L} Tr(t_1 t_1 t_1 t_2 t_2 t_1 t_1 \ldots t_2) \end{aligned}$$

Wish: Systematic approach to the computation of 1-pt functions of *conformal* operators.

“Pure” AdS/CFT

Local gauge invariant conformal operators $\longleftrightarrow$ string states

Conformal dimensions Δ $\longleftrightarrow$ energies of string states

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta'}(0)\rangle = \delta_{\Delta,\Delta'}\,\frac{1}{|x|^{2\Delta}}, \qquad \langle \mathcal{O}_\Delta(x)\rangle = 0$$

Determine $\Delta's$ and $\mathcal{O}_\Delta$'s in the CFT $\equiv$

Solve the spectral problem $\equiv$

Diagonalize dilatation operator $\hat{D}$

In the planar limit, $N \to \infty$:

$\hat{D} = \hat{H}$ of an integrable spin chain

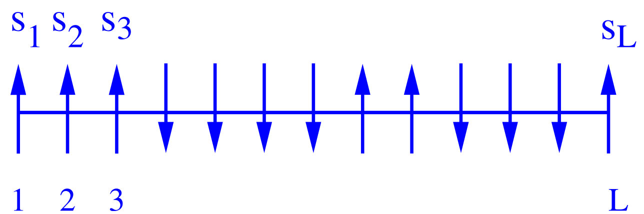
In the planar limit, $N \to \infty$:

The SU(2) sub-sector

Further simplifications:

- One-loop level
- Restriction to $SU(2)$-sector $\subset PSU(2,2|4)$

$\Rightarrow \ \hat{D} = \hat{H}_{heisenberg}$



$S_{L+m} = S_m$

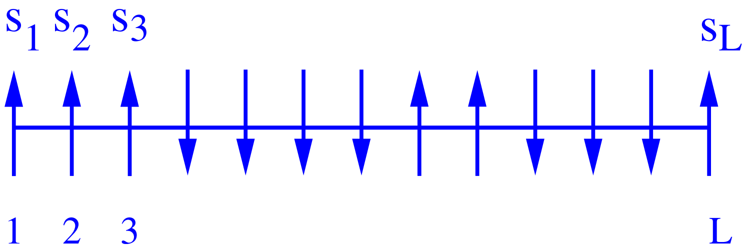
$$\mathcal{O} = \mathrm{Tr}(\underbrace{Z\,Z\,ZW\,W\,Z\,Z\ldots W}_{\text{L fields}}) \sim |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\ldots\downarrow\rangle$$

$$\hat{\mathcal{D}}\,\mathcal{O}_\Delta = \Delta\,\mathcal{O}_\Delta$$

In the presence of the defect

$$\langle\mathcal{O}_\Delta(z)\rangle = \frac{C}{|z|^\Delta}, \quad z<0, \qquad \langle\mathcal{O}_\Delta(z)\rangle = 0, \quad z>0$$

The integrable Heisenberg spin chain



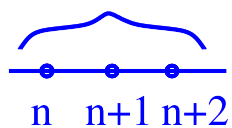
$$S_{L+m} = S_m$$

$$\hat{H} = -\lambda \sum_{n=1}^{L} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} = \lambda \sum_{n=1}^{L} (1 - P_{n,n+1}) = \sum_{n=1}^{L} \hat{H}_{n,n+1}$$

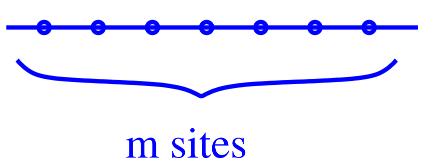
Conserved charges: $\exists\ \hat{Q}_i, \quad i = 1, \ldots L : \quad [\hat{Q}_i, \hat{Q}_j] = 0$

$$\hat{Q}_1 = \sum_n e^{i\hat{P}_n}, \quad \hat{Q}_2 = \hat{H}$$

$$\hat{Q}_3 = \sum_n [\hat{H}_{n,n+1}, \hat{H}_{n+1,n+2}] =$$



$\hat{Q}_m$:



Eigenstates of the Heisenberg spin chain

Ground State $(\lambda > 0)$: $\hat{H}|\uparrow\uparrow\ldots\uparrow\rangle \equiv \hat{H}|0\rangle = 0$

Excited states (with M flipped spins):

$$|\{u_i\}\rangle = \hat{B}(u_M)\ldots\hat{B}(u_1)|0\rangle$$

where

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j=1, j\neq k}^{M} \left(\frac{u_k - u_j + i}{u_k - u_j - i}\right), \quad k = 1, \ldots M$$

with

$u_i = \frac{1}{2}\cot\left(\frac{p_i}{2}\right)$ where p_i are momenta

Interested in zero-momentum sector (= cyclically inv. states)

$$\sum_{i=1}^{M} p_i = 0$$

Paired and un-paired solutions

Bethe eqns invariant under $\{u_k\} \to \{-u_k\}$

Solutions can be split into paired and unpaired:

Paired solutions:

$$|\{u_k\}\rangle, |\{-u_k\}\rangle, \quad \text{where} \quad \{u_k\} \neq \{-u_k\}$$

$$Q_{2n+1}|\{u_k\}\rangle \neq 0, \;\; n = 1, 2, \ldots$$

Unpaired solutions:

$$|\{u_k\}\rangle, \quad \text{where} \quad \{u_k\} = \{-u_k\}$$

$$Q_{2n+1}|\{u_k\}\rangle = 0, \;\; n = 1, 2, \ldots$$

Tree-level one-point functions

$$\langle O_\Delta \rangle \sim \text{“}\langle \{u_i\} \rangle\text{”}$$

Wish: Systematic approach to the computation of 1-pt functions of Bethe eigenstates

Matrix Product State associated with the defect:

$$\langle MPS| = \mathrm{tr}_a \prod_{l=1}^{L} \left(\langle \uparrow_l | \otimes t_1 + \langle \downarrow_l | \otimes t_2 \right)$$

Object to calculate: $C\left(\{u_j\}\right) = \dfrac{\langle MPS\,|\{u_j\}\rangle}{\langle \{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}}$

where $|\{u_j\}\rangle = B(u_1)\ldots B(u_M)\,|0\rangle_L$

is a Bethe eigenstate of length L having M excitations.

Tree-level one-point functions of Bethe eigenstates

Dream scenario:

$C\left(\{u_j\}\right)$ given by closed expression of determinant type.

Motivation: Several examples of this exist

- Gaudin formula for norm of Bethe eigenstate (i.e. on-shell state).
- Slavnov formula for inner product between an on-shell and an off-shell Bethe state
- Determinant formula for inner product of Bethe on-shell state with the Néel state. [Pozsgay '13, Brockmann et al '14]

$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow\ldots\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\ldots\downarrow\uparrow\rangle$$

OBS: The defect state is not a Bethe state! (And not the Néel state)

Strategy of calculation

Matrix Product State indicates the usefulness of the algebraic BA.

$$\langle MPS| = \mathrm{tr}_a \prod_{l=1}^{L} \left(\langle \uparrow_l | \otimes t_1 + \langle \downarrow_l | \otimes t_2 \right)$$

$$= \langle \uparrow \ldots \uparrow | \, K, \qquad \text{where}$$

$$K = \mathrm{tr}_a \prod_{l=1}^{L} \left\{ \left[s \cdot 1 + (1-s) \sigma_l^3 \right] \otimes t_1 + \sigma_l^+ \otimes t_2 + \sigma_l^- \otimes t \right\},$$

$$= \mathrm{tr}_a \prod_{l=1}^{L} \left(\sigma_l^3 \otimes t_1 + \sigma_l^1 \otimes t_2 \right), \qquad \text{for} \qquad s = 0, \; t = t_2$$

Reminiscent of the ABA construction

But so far calculations are done by the coordinate space BA

$$|\{p_i\}\rangle = \sum_{\sigma \in S_M} \sum_{1 \le n_1 < \ldots < n_M \le L} e^{i \sum_m \left(p_{\sigma_m} n_m + \sum_{j<m} \frac{\theta_{\sigma_j \sigma_m}}{2} \right)} S_{n_1}^- \ldots S_{n_M}^- |0\rangle,$$

with $u_i = 1/2 \cot(p_i/2)$

Encodes the S-matrix

General results

- The overlap vanishes unless M and L are both even

  (Easy to see for $k = 2$ where $\{t_i, t_j\} = 0$, but true for any k.)

- The overlap vanishes unless the Bethe eigenstate has $P_{tot} = 0$

  Follows from the fact that $|MPS\rangle$ has $P_{tot} = 0$

$$\left(\langle MPS \,\middle|\, U\right) \,\middle|\, \{u_j\}\rangle = \langle MPS \,\middle|\, \{u_j\}\rangle = \langle MPS \,\middle|\, \left(U \,\middle|\, \{u_j\}\rangle\right),$$

where $U = e^{i\hat{P}_{tot}}$

- The overlap vanishes except for unpaired states: $\{u_i\} = \{-u_i\}$

  Follows from the fact that $Q_3|MPS\rangle = 0$, paired states have $q_3 \neq 0$ and

$$0 = \langle MPS|Q_3|\Psi\rangle = q_3\langle MPS|\Psi\rangle$$

More specific results (can be obtained ``by hand”)

1. Overlap with the vacuum (M=0, any k and L):

$$\langle \Psi^{\rm cl}|0\rangle = {\rm tr}\, {\rm t}_3^{\rm L} = \zeta_{-\rm L}\left(\tfrac{1-{\rm k}}{2}\right) - \zeta_{-\rm L}\left(\tfrac{1+{\rm k}}{2}\right) = \frac{{\rm k}^{\rm L+1}}{2^{\rm L}({\rm L}+1)} + \mathcal{O}(k^L)$$

2. Two excitations (M=2, any k and L)

$$\langle \Psi^{\rm cl}|p,-p\rangle = Lu(u-\tfrac{i}{2}) \sum_{j=-\frac{k}{2}}^{\frac{k}{2}} \frac{j^2-\frac{k^2}{4}}{j^2+u^2}(j-\tfrac{1}{2})^{L-1}$$

For k=2:

$$\langle \Psi^{\rm cl}|p,-p\rangle = 2^{1-L}\, Lu^{-1}(u-\tfrac{i}{2})$$

For large k:

$$\langle \Psi^{\rm cl}|p,-p\rangle = \frac{u(u+\frac{i}{2})}{L-3}\,\frac{k^{L-1}}{2^L} + \frac{u(u+\frac{i}{2})}{(L-1)(L-3)}\,\frac{k^{L-2}}{2^L} + \mathcal{O}(k^{L-3})$$

Result for k=2, any M, L (obtained using Mathematica)

$$|\{u_j\}\rangle = B(u_1)B(-u_1)\dots B(u_{\frac{M}{2}})B(-u_{\frac{M}{2}})\,|0\rangle$$

$$C_2(\{u_j\}) = \frac{\langle MPS \mid \{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}} = 2^{1-L}\left(\prod_j \frac{u_j^2+\frac{1}{4}}{u_j^2}\;\frac{\det G^+}{\det G^-}\right)^{\frac{1}{2}}$$

$$G^{\pm}_{jk} = \left(\frac{L}{u_j^2+\frac{1}{4}} - \sum_n K^+_{jn}\right)\delta_{jk} + K^{\pm}_{jk}.$$

$$K^{\pm}_{jk} = \frac{2}{1+(u_j-u_k)^2} \pm \frac{2}{1+(u_j+u_k)^2}\,,$$

All matrices of size M/2 x M/2

Proof of the formula for k=2 (M=L/2)

- Proposal based on explicit (Mathematica) calculations up to and including M=8, L=20

- Formula can be proved for M=L/2

Observation: $$C(\{u_j\}) = \frac{1}{4^M \left(\frac{i}{2}\right)^{\frac{M}{2}}} \cdot \frac{\langle \text{Néel} \,|\{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}} \,,$$

$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow\ldots\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\ldots\downarrow\uparrow\rangle$$

$$|\Psi^{\text{cl}}\rangle = \frac{1}{4^M \left(\frac{i}{2}\right)^M} |\text{Néel}\rangle + S^- |\ldots\rangle \,, \qquad S^+ |\{u_j\}\rangle = 0$$

Result for any k, M, L

$$C_k\left(\{u_j\}\right) = k^{L-2M+1}\left[\sum_{j=1-\frac{k}{2}}^{\frac{k}{2}} \frac{1}{2k}\left(\frac{2j-1}{k}\right)^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2\left(\frac{u_i^2}{k^2}+\frac{1}{4}\right)}{\left(\frac{u_i^2+j^2}{k^2}\right)\left(\frac{u_i^2+(j-1)^2}{k^2}\right)}\right] C_2\left(\{u_j\}\right).$$

The limit $k \to \infty$ is of interest to string theory

Connection to string theory

The AdS/dCFT set-up: Extra parameter k

Field theory side: dimension of rep. of vev of scalars

String theory side: Number of D3 branes dissolved into D5 brane

First take the planar limit: $N \to \infty,\ g_s \to 0$

Next consider $\lambda \to \infty, k \to \infty,\ \frac{\lambda}{k^2}$ finite (BMN like limit)

Comparisons can be made order by order in $\frac{\lambda}{k^2}$

Match found to leading order in $\frac{\lambda}{k^2}$ in for chiral primaries

Nagasaki & Yamaguchi '12, C.K, Semenoff & Young '12

Comparison with string theory

Agreement found to leading order in $\frac{\lambda}{k^2}$
for operators which are chiral primaries (protected in theory without defects).

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{k}{\sqrt{\Delta}} \left(\frac{2\pi^2 k^2}{\lambda} \right)^{\Delta/2} Y_\Delta(0) \, \frac{1}{|z|^\Delta},$$

Field theory side: Calculated by insertion of vev in spherical harmonics
with the appropriate symmetry.

String theory side: Calculated using the supergravity approximation
(Fluctuation of D5 brane action when an appropriate source is inserted on the boundary of AdS)

Nagasaki & Yamaguchi '12, C.K, Semenoff & Young '12

Open questions

- Proof of the k=2 determinant formula for $M \neq L/2$
  (work in progress)
- Proof of the determinant formula for general k, L, M
  (work in progress)
- Consider the thermodynamical limit $M, L \to \infty,\ M/L$ finite
  (work in progress)
- Higher loops, other sectors in the dCFT
- Other dCFT's/ other probe brane set-ups such as D3-D7
- More detailed comparisons with string theory:
  f.inst. involving spinning strings
  (work in progress)

Conclusion

- The dream scenario was realized
- The tools of integrability came in handy
- Many interesting open questions remain