# Entanglement in Spin Chains

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**Entanglement is a resource for quantum computation. How much quantum effects we can use to control one quantum system by another.**

## OUTLINE

**1. Entanglement of two quantum subsystems $A$ and $B$. Simplest case when the whole system $\{A, B\}$ is in a pure state: a unique wave function $|\Psi^{A,B}\rangle$. Entropy of a subsystem is a measure of entanglement.**

**2. The wave function will be a unique ground state of a dynamical model: interacting spins, Bose gas or strongly correlated electrons. Universal properties of entropy of a large subsystem.**

Easy to explain when entanglement is absent:

$$|\Psi^{A,B}\rangle = |\Psi^{A}\rangle \otimes |\Psi^{B}\rangle \Longleftrightarrow \text{no entanglement}$$

But if the wave function is a sum of several such terms

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{d} \quad |\Psi_j^{A}\rangle \otimes |\Psi_i^{B}\rangle \qquad \text{entangled.}$$

Here $d > 1$ and $|\Psi_j^A\rangle$ are linear independent; as well as $|\Psi_j^B\rangle$.

Measure of entanglement? C.H. Bennett, H.J. Bernstein,

S.Popescu, and B.Schumacher 1996 entropy of a subsystem :

$$S = -tr_A\left(\rho_A \ln \rho_A\right), \qquad \rho_A = tr_B\left(|\Psi^{A,B}\rangle\langle\Psi^{A,B}|\right)$$

**Contra-intuitive features: Two spins $1/2$ forming spin $1$.**

$$|\uparrow\rangle \otimes |\uparrow\rangle \qquad \text{no entanglement}$$

**but middle component with $S_z = 0$ is maximally entangled:**

$$|\uparrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle \qquad \text{maximum entanglement}$$

**Entanglement depends on $S_z$.**

**Ferromagnetic XXX:**

$$\mathrm{H_f} = -\sum_{j=1}^{L} \{\boldsymbol{\sigma}_j^x \boldsymbol{\sigma}_{j+1}^x + \boldsymbol{\sigma}_j^y \boldsymbol{\sigma}_{j+1}^y + \boldsymbol{\sigma}_j^z \boldsymbol{\sigma}_{j+1}^z\}$$

**$\boldsymbol{\sigma}$ are Pauli matrices. $[\mathrm{H_f}, S_z] = 0$, $S_z = \left(\sum_{j=1}^{L} \boldsymbol{\sigma}_j^z\right)$**

**We can fix $S_z$ to fix ground state uniquely:**

**Maximum of $S_z = \left(\sum_{j=1}^{L} \boldsymbol{\sigma}_j^z\right)$**

$$|\uparrow\rangle \otimes |\uparrow\rangle \ldots \otimes |\uparrow\rangle \qquad \text{no entanglement.}$$

We can lower $S_z$ by applying the operator $S^- = \left(\Sigma_{j=1}^{L} \sigma_j^-\right)$:

$$|\Psi_{fm}\rangle = (S^-)^M \left(|\uparrow\rangle \otimes |\uparrow\rangle \ldots \otimes |\uparrow\rangle\right) \qquad \text{entangled}$$

Non-trivial magnetization $S_z/L$ if $M \to \infty$ proportionally to $L \to \infty$. We shall artificially represent the ground state as by partite system:

Block of spins on an space interval $[1, x]$ is the subsystem $A$ ,

the rest of the ground state is the subsystem $B$.

Entanglement of a block of spins on a space interval $[1, x]$ with the rest of the ground state $|\Psi_{fm}\rangle$.

Density matrix of the block $\rho(x)$ the entropy of the block $S(x)$.

$$S(x) \to \frac{1}{2} \ln(x), \qquad x \to \infty, \qquad \text{F}$$

Double scaling limit $1 << x << L$. Salerno and Popkov 2004

## Anti - ferromagnetic XXX:

$$\mathrm{H}_{\mathrm{af}} = \sum_j \{\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z\}$$

**The ground state is unique and has spin $0$. Also a gapless case, but for low lying excitations energy is proportional to momentum $\epsilon \sim p$. So one can use conformal field theory for evaluation of asymptotic of entropy of a large block of spins, belonging to the ground state:**

$$S(x) \to \frac{1}{3}\ln(x), \qquad x \to \infty, \qquad \textbf{AF}$$

**Holzhey, Larsen and Wilczek 1994. Finite size corrections**

$$S(x) = \ln(x)/3 + x_0 + \sum_{j=1}^{\infty} a_j/x^j$$

**Zamolodchikov, Fatteev, Takhatajan and Babujian:**

$$\mathrm{H}_1 = \sum_n \{X_n - X_n^2\} \qquad \textbf{spin} \qquad \mathrm{s} = 1$$

$$X_n = \vec{S}_n \vec{S}_{n+1} = S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z$$

**Solvable by Bethe Ansatz.**

**Higher spin s Faddeev 1983:**

$$\mathrm{H}_s = \sum_n F(X_n), \qquad F(X)\textbf{is a polynomial of degree } 2\mathrm{s}$$

$$F(X) = 2 \sum_{l=0}^{2\mathrm{s}} \sum_{k=l+1}^{2\mathrm{s}} \frac{1}{k} \prod_{\substack{j=0 \\ j \neq l}}^{2\mathrm{s}} \frac{X - y_j}{y_l - y_j}, \qquad y_l = l(l+1)/2 - \mathrm{s}(\mathrm{s}+1)$$

**The entropy of a block of $x$ spins V.K. 2003:**

$$S\,(x) = \frac{\mathrm{s}}{\mathrm{s}+1} \ln x, \quad \textbf{as} \quad x \to \infty$$

Other models: Bose gas with delta interaction:

$$H = \int dx \left[\partial\psi_x^\dagger \partial\psi_x + g\psi^\dagger\psi^\dagger\psi\psi\right].$$

Here $\psi$ is a canonical Bose field and $g > 0$ is a coupling constant.

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j} + 2g \sum_{N \geq k > l \geq 1} \delta(x_k - x_l)$$

Bethe Ansaz: E. Lieb, W.Liniger in 1963.

The entropy of gas on a space interval $[0, x]$ also scales as

$$S(x) \to \frac{1}{3}\ln(x), \qquad x \to \infty$$

V.K, 2004

The Hubbard model $H$:

$$H = - \sum_{\substack{j=1 \\ \sigma=\uparrow,\downarrow}} (c^\dagger_{j,\sigma} c_{j+1,\sigma} + c^\dagger_{j+1,\sigma} c_{j,\sigma}) + u \sum_{j=1} n_{j,\uparrow} n_{j,\downarrow}$$

Here $c^\dagger_{j,\sigma}$ is a canonical Fermi operator on the lattice [creates of an electron] and $n_{j,\sigma} = c^\dagger_{j,\sigma} c_{j,\sigma}$ is an operator on number of electrons in cite number $j$ with spin $\sigma$, $u > 0$. Lieb and Wu in 1968. Below half filling [less then one electron per lattice cite] both charge and spin degrees of freedom can be described by Virasoro algebra with central charge equal to 1, Korepin, Frahm 1990 : $c_c = 1$ and $c_s = 1$.

$$S(x) = \frac{2}{3} \ln x$$

V.K. 2004. At half filed band $S(x) = (1/3) \ln x$

## Rényi entropy also scales logarithmically

$$S_{\alpha} = \frac{1}{1-\alpha} \ln Tr(\rho_A^{\alpha}) = \left(\frac{1+\alpha^{-1}}{6}\right) \ln x$$

**Bai Qi Jin , Korepin 2004**

**The entropy of a block of spins for a model with a gap approaches a constant $S_\infty$ as the size of the block increases:**

$$S(x) \to S_\infty \qquad \text{as} \qquad x \to \infty$$

**The Hamiltonian of $XY$ model :**

$$H = -\sum_{n=-\infty}^{\infty} (1+\gamma)\sigma_n^x \sigma_{n+1}^x + (1-\gamma)\sigma_n^y \sigma_{n+1}^y + h\sigma_n^z$$

**Here $\gamma$ is anisotropy parameter: $0 < \gamma < 1$ and $0 < h$ is a magnetic field and $\sigma_n$ are Pauli matrices. Solution, phase transitions: Lieb, Schultz, Mattis, Barouch and McCoy. Toeplitz determinants and integrable Fredholm operators were used for evaluation of correlation functions,**

**The model has three different cases:**

1. Case 1**a:** $2\sqrt{1-\gamma^2} < h < 2.$ **medium magnetic field**
2. Case 1**b:** $0 \leq h < 2\sqrt{1-\gamma^2}$ **, small magnetic field**
3. Case **2:** $h > 2$ **, strong magnetic field**

**A.Its, B.-Q. Jin, V. Korepin 2004: Case 1b**

$$S_\infty = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + \left(1 - \frac{k^2}{2}\right)\frac{4I(k)I(k')}{\pi}\right] + \ln\ 2$$

**with** $k' = \sqrt{1-k^2}$**, and**

$$k = \sqrt{\frac{1-(h/2)^2-\gamma^2}{1-(h/2)^2}} \qquad I(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

$I(k)$ **is the complete elliptic integral of the first kind:**

## I. Peschel 2004: Case 1a: medium magnetic field

$$S_\infty = \frac{1}{6}\left[\ \ln\left(\frac{k^2}{16k'}\right) + \left(1 - \frac{k^2}{2}\right)\frac{4I(k)I(k')}{\pi}\right] + \ln\ 2,$$

**and in** Case 2 **: strong magnetic field**

$$S_\infty = \frac{1}{12}\left[\ \ln\frac{16}{(k^2k'^2)} + (k^2 - k'^2)\frac{4I(k)I(k')}{\pi}\right], \tag{1}$$

$$k = \begin{cases} \sqrt{(h/2)^2 + \gamma^2 - 1}\ /\ \gamma, & \textbf{Phase 1a} \\ \gamma\ /\ \sqrt{(h/2)^2 + \gamma^2 - 1}, & \textbf{Phase 2} \end{cases} \tag{2}$$

**Consider range of variation: $S_\infty$ reaches minimum as ordered states and it reaches maximum at phase transitions [disordered states].**

Local minimum $S_\infty = \ln 2$ at boundary between cases 1a and 1b: $h = 2\sqrt{1-\gamma^2}$

The ground state is doubly degenerated:

$$|G_1\rangle = \prod_{n\in\text{lattice}} [\cos(\theta)|\uparrow_n\rangle + (-1)^n \sin(\theta)|\downarrow_n\rangle]$$

$$|G_2\rangle = \prod_{n\in\text{lattice}} [\cos(\theta)|\uparrow_n\rangle - (-1)^n \sin(\theta)|\downarrow_n\rangle]$$

Each state is factorized and has no entropy.

$\cos^2(2\theta) = (1-\gamma)/(1+\gamma)$

Absolute minimum of is reached at $h \to \infty$ : $S_\infty \to 0$ as the state becomes ferromagnetic $\uparrow \ldots \uparrow$.

These are most ordered states. These is a gap.

## Phase transitions: as the gap closes $S_\infty \to +\infty$

**1.** *Critical magnetic field*: $h \to 2$ **and** $\gamma \neq 0$:

$$S_\infty \to -\frac{1}{6}\ln|2-h| + \frac{1}{3}\ln 4\gamma + O(|2-h|\ln^2|2-h|)$$

**2.** *An approach to $XX$ model*: $\gamma \to 0$ **and** $0 < h < 2$:

$$S_\infty \to -\frac{1}{3}\ln\gamma + \frac{1}{6}\ln(4-h^2) + \frac{1}{3}\ln 2 + O(\gamma\ln^2\gamma)$$

**This agrees with conformal approach of P. Calabrese, J. Cardy and Toeplitz determinant approach of B.-Q. Jin, V.Korepin.**

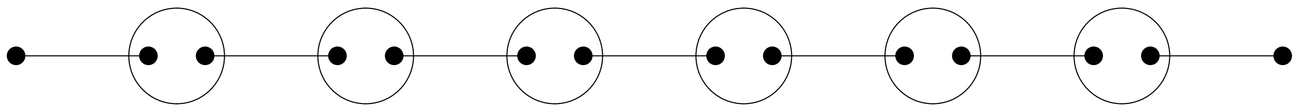
Let us consider another gapped spin chain introduced by Affleck, Kennedy, Lieb, and Tasaki: AKLT model also known as VBS. It consists of a chain of $N$ spin-1's in the bulk, and two spin-1/2 on the boundary. We shall denote by $\vec{S}_k$ the vector of spin-1 operators and by $\vec{s}_b$ spin-1/2 operators at boundaries.

$$H = \sum_{k=1}^{N-1} \left( \vec{S}_k \vec{S}_{k+1} + \frac{1}{3} (\vec{S}_k \vec{S}_{k+1})^2 \right) + \pi_{0,1} + \pi_{N,N+1}.$$

Notice that $\frac{1}{2}\vec{S}_k \vec{S}_{k+1} + \frac{1}{6}(\vec{S}_k \vec{S}_{k+1})^2 + \frac{1}{3}$ is a projector on a state of spin $2$. The terms $\pi$ describe interaction of boundary spin 1/2 and next spin 1; $\pi$ is a projector on a state with spin 3/2:

$$\pi_{0,1} = \frac{2}{3}\left(1 + \vec{s}_0 \vec{S}_1\right), \quad \pi_{N,N+1} = \frac{2}{3}\left(1 + \vec{s}_{N+1} \vec{S}_N\right).$$

**Ground state is unique and there is a gap:**



**a dot is spin-$\frac{1}{2}$; circle means symmetrisation [makes spin $1$]. A line is a anti-symmetrisation $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ lower joint spin.**

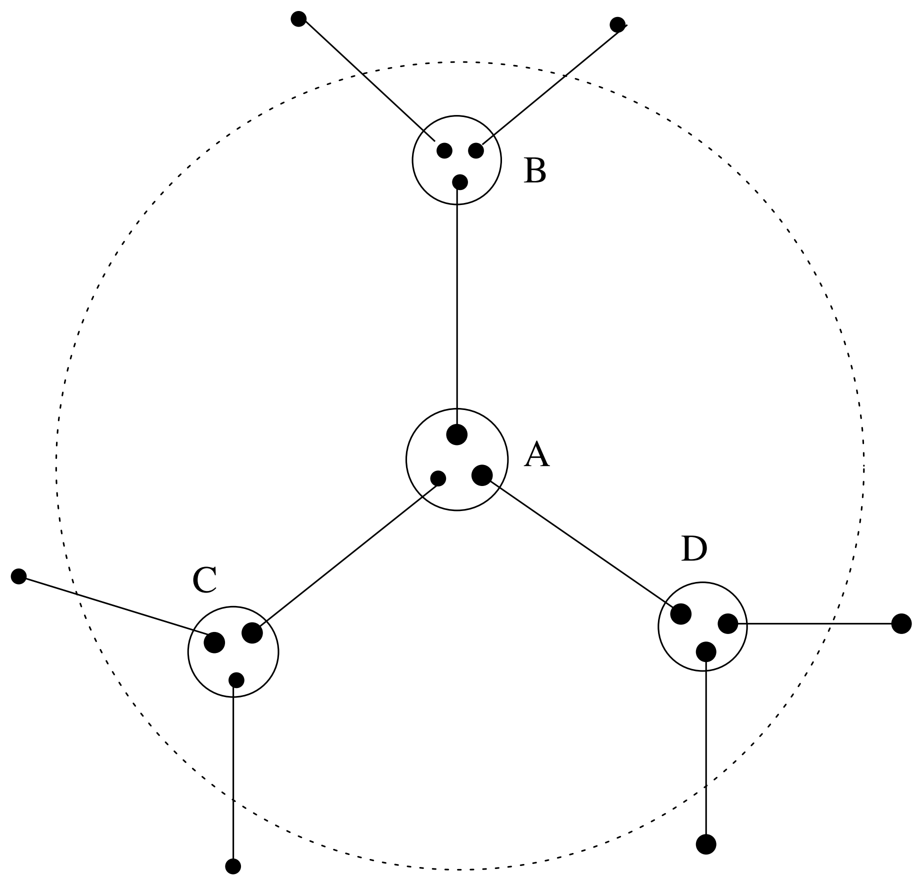
**Correlation function are: $< \vec{S}_x \vec{S}_1 > \sim (-1/3)^x = p(x)$**

**Fan, Korepin and Roychowdhury calculated entropy of a finite block of spins in the ground state on a finite lattice:**

$$S(x) = 2 + \frac{3(1-p(x))}{4} \log\left(1 - p(x)\right) - \\ - \frac{1+3p(x)}{4} \log\left(1 + 3p(x)\right),$$

**In double scaling limit the expression simplifies:**

$$S(x) \longrightarrow 2 - (3/2)p(x) \qquad \textbf{as} \qquad x \longrightarrow \infty$$




**AKLT construction is universal. Consider Cayley tree: each lattice site has three neighbors; no loops. Spin at a lattice site is $3/2$. Each dot is spin $1/2$, circle is symmetrisation, line is anti-symmetrisation.**

The Hamiltonian of the AKLT model is

$$H = \sum_{(i,j)} P_3(\vec{S}_i + \vec{S}_j) \qquad (3)$$

$P_3$ is a projector on a joint state with spin $3$, and $(i, j)$ are neighbors on the lattice. The ground state is unique and similar to 1D case. Symmetrisation in each lattice cite and anti-symmetrisation along the links. There is a gap. We calculated the entropy of a block of spins:

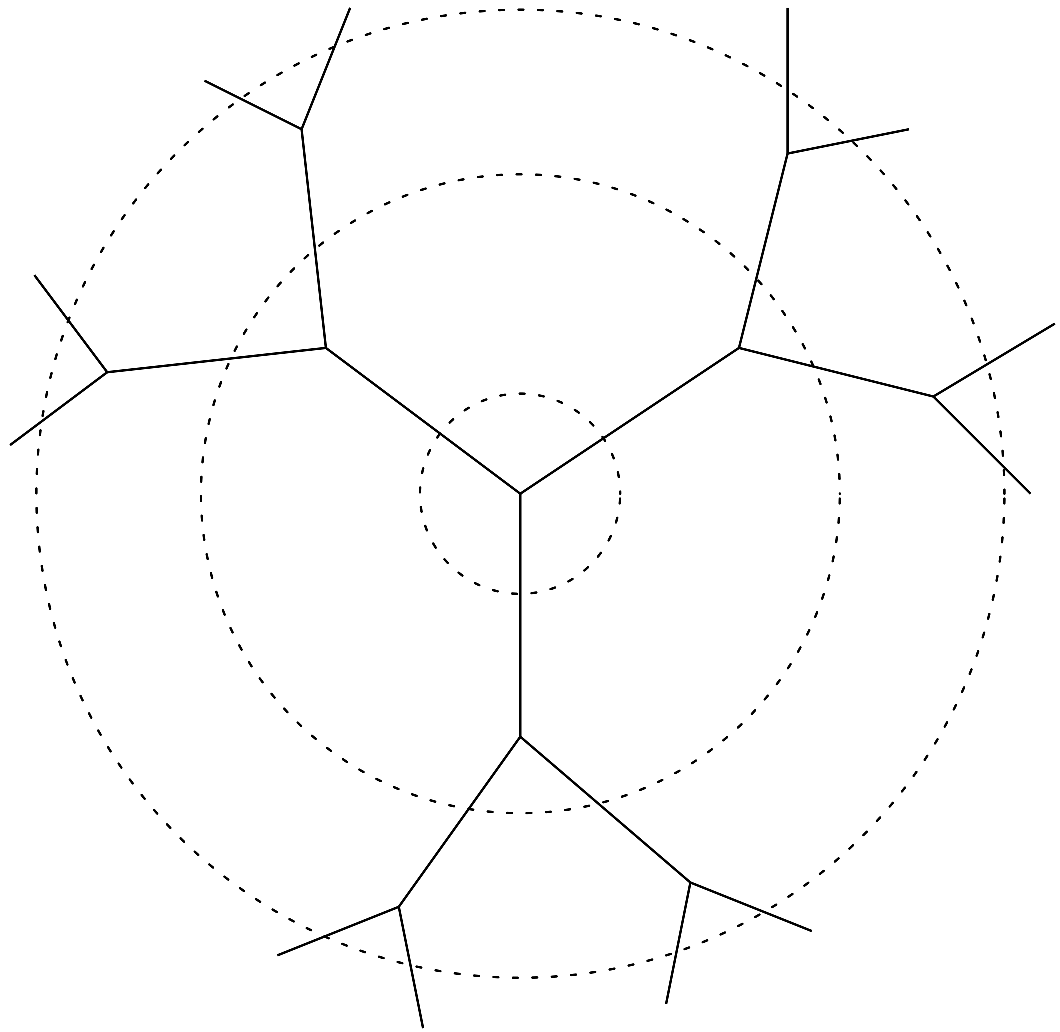


Figure 1: Cayley tree has no loops. Dotted circle shows the block of spins.

$\boldsymbol{S_\infty}$ = number of links necessary to cut to isolate the block.

**Up to now we considered only pure states: unique ground state.**

**How we measure the entanglement in mixed states?**

**First in models described by conformal field theory one can calculate the entropy of a block of spins for positive temperature:**

$$S(x) = \frac{1}{3} \ln \left( \frac{v_s}{\pi T} \sinh \left[ \frac{\pi T x}{v_s} \right] \right)$$

**V.K. 2003. For large blocks**

$$S(x) = \frac{\pi T x}{3 v_s} \qquad \textbf{as} \qquad x \to \infty$$

**This is second law of thermodynamics. At large temperature the block of spins strongly interacts with the environment [rest of the ground state] off-diagonal elements of density matrix vanish [de-coherence]. The block turns into a classical system of macroscopical size. Lot of entropy, but no quantum effects.**

For mixed states entropy of a subsystem is not a measure of entanglement. What is a measure of entanglement of two quantum subsystems $A$ and $B$ which stay together in a mixed state $\{A, B\}$? We can describe the state $\{A, B\}$ by a density matrix $\rho_{A,B}$. In 2001G. Vidal and R.F. Werner suggested negativity as measure of entanglement. Recall that $\rho_{A,B}$ is a positive matrix. But partially transposed $\rho_{A,B}^{T_A}$ does not have to be positive. A measure of entanglement of two quantum subsystems $A$ and $B$ in a mixed state $\{A, B\}$ is negativity:

$$\mathcal{N}_{A,B} = |\,\Sigma\, \text{negative eigenvalues of } \rho_{A,B}^{T_A}|$$

Vidal and Werner proved that it vanishes for unentangled states. $\mathcal{N}_{A,B}$ does not increase under LOCC.

**Negativity does not increase under local manipulations of the system. It provides a bound on the teleportation capacity and on the distillable entanglement of mixed states. Negativity can be used to quantify a degree of entanglement in mixed systems. Sometimes logarithmic negativity also used**

$$\mathcal{E} = \log_2 \left(2\mathcal{N}(\rho) + 1\right)$$

## SUMMARY

**The entanglement has to be measured differently for pure and mixed states. Entropy of a subsystem measures entanglement at zero temperature only. In double scaling limit it shows universality. For gapped models entropy of a subsystem is proportional to the area of the boundary. In 1D many gapless models can be described by conformal field theory. Still one should be careful, for some gapless models $\epsilon \sim p^2$, then the entropy also scales logarithmically but the coefficient is different. Analysis of entropy helps to find most ordered states.**

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