Topological Entanglement Entropy from the Holographic Partition Function

There are very deep connections between topological states in 2+1 dimensions, chiral conformal field theories in 1+1 dimensions, and quantum impurity problems.

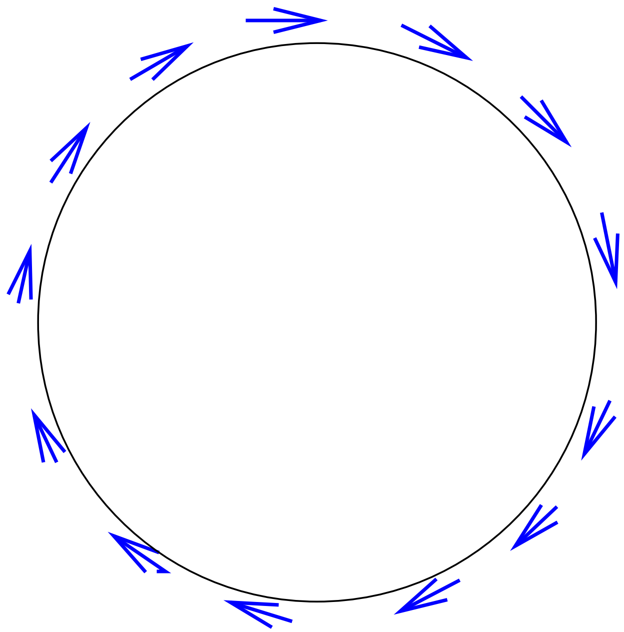
Outline:

1. Entanglement and quantum dimension
2. Topological entanglement entropy
3. The edge
4. Holographic partition functions
5. Disentanglement by a point contact

work with Matthew Fisher and Chetan Nayak

The setting:

Quantum states in two dimensions with gapless chiral edge modes.



Simple examples are Laughlin quantum Hall states, where the edge modes are described by free chiral bosons in 1+1 dimensions.

The two key examples for this talk are:

- a $p_x + ip_y$ superconductor

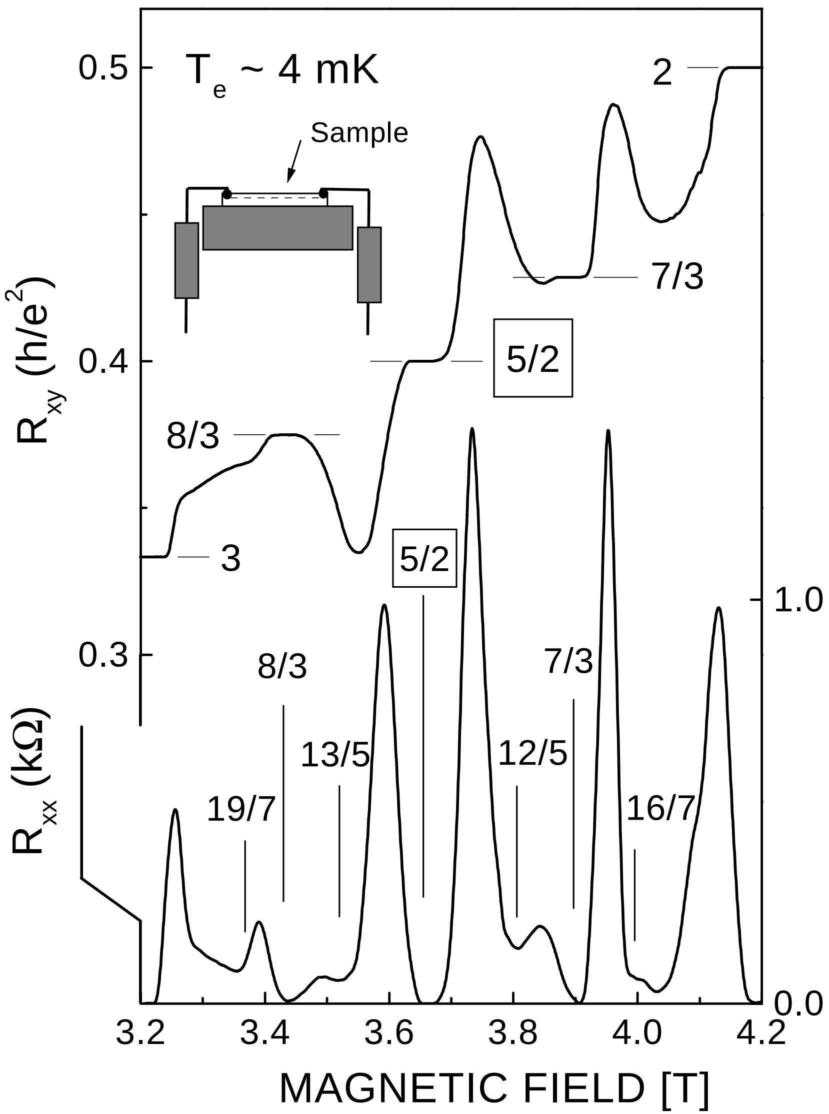
  Read and Green; Ivanov; Stern et al; Chung and Stone

- the Moore-Read state for the $\nu = 5/2$ fractional quantum Hall effect

For field theorists: think Chern-Simons theory on a disk $\times$ time.

With pure enough samples:

Fig. 1, Pan et al



data from Pan et al

The only theory of the $5/2$ plateau consistent with the data and the numerics predicts charge $e/4$ particles with non-abelian statistics.

Moore and Read

If you could manipulate these quasiparticles, you could build a topological quantum computer.

But that's another talk...

A convenient way of labeling the excitations of such theories is by topological sector.

In the Hall effect, this is mod an electron. Any states which are related to each other by adding an electron or hole are identified.

Thus e.g. for the $\nu = 1/3$ abelian Laughlin state, the three sectors are labeled by the charges $0, 1/3, -1/3$.

The sectors for the $p + ip$ superconductor are

- I (the electron in the MR state)
- ψ (neutral fermion)
- σ (the interesting one!)

  in $p + ip$ this creates a vortex

For Moore-Read, one must include (abelian) charge degrees of freedom as well.

Non-abelian statistics means that the system changes state when quasiparticles are brought around each other.

For this to be possible, there must be degeneracies in multi-quasiparticle Hilbert spaces.

These degeneracies can be found from the fusion rules.

For $p + ip$ or $\nu = 5/2$, the non-trivial fusion rule is

$$\sigma \cdot \sigma = I \; + \psi$$

There are two possible quantum states for two σ particles, no matter how far apart they are.

Entanglement!

The full set of fusion rules for $p + ip$ is

$$\begin{aligned}
\sigma \cdot \sigma &= I + \psi \\
\sigma \cdot \psi &= \sigma \\
\psi \cdot \psi &= I
\end{aligned}$$

The only time there is more than one thing on the right-hand-side is for $\sigma \cdot \sigma$.

The fusion algebra allows us to count the number of quantum states for $2N$ quasiparticles:

$$\underbrace{\sigma \quad \sigma}_{I,\psi} \quad \underbrace{\sigma \quad \sigma}_{I,\psi} \quad \underbrace{\sigma \quad \sigma}_{I,\psi} \quad \underbrace{\sigma \quad \sigma}_{I,\psi} \quad \underbrace{\sigma \quad \sigma}_{I,\psi}$$

There are thus 2^N states for $2N$ quasiparticles.

The entropy per particle is

$$\frac{\ln(\text{\# of states})}{2N} = \ln(\sqrt{2})$$

The quantum dimension of σ is

$$d_\sigma = \sqrt{2}$$

The quantum dimensions of the other fields are $d_I = 1$, $d_\psi = 1$.

Non-abelian statistics is possible only when $d > 1$.

Topological entanglement entropy

The von Neumann entropy measures the entanglement between two subregions of a system. On general grounds, for a gapped and local theory in two dimensions this grows as the length L of the boundary:

$$S_{VN} = aL - \gamma + \dots$$

In a topological state this subleading piece is universal and very interesting.

The (universal part of the) topological entanglement entropy of a disc with no bulk quasiparticles is

$$-\gamma = -\ln \mathcal{D}$$

where $\mathcal{D}$ is the total quantum dimension

$$\mathcal{D} = \sqrt{\sum (d_a)^2}$$

Kitaev and Preskill, Levin and Wen

The sum is over all the topological sectors (i.e. in the FQHE, all sectors mod an electron).

For the $\nu = 1/3$ Laughlin state:

$$\begin{aligned}\mathcal{D} &= \sqrt{(d_0)^2 + (d_{1/3})^2 + (d_{-1/3})^2} \\ &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}\end{aligned}$$

For $p + ip$:

$$\begin{aligned}\mathcal{D} &= \sqrt{(d_I)^2 + (d_\psi)^2 + (d_\sigma)^2} \\ &= \sqrt{1^2 + 1^2 + (\sqrt{2})^2} \\ &= 2\end{aligned}$$

Say we insert a point contact in the middle. In the field theory, this is a relevant perturbation. Thus if the temperature is low or the gate voltage high, the system is effectively split into two.

Without the point contact (and no bulk quasiparticles), $\gamma_{UV} = -\ln \mathcal{D}$.

In the IR, the system has split into two discs, so $\gamma_{IR} = -2\gamma_{UV}$.

Thus we have an entropy loss:

$$\gamma_{UV} - \gamma_{IR} = -\gamma_{UV}$$

A point contact causes disentanglement!

But what is being disentangled?

The edge

In these theories, bulk excitations are gapped, but edge modes are gapless (c.f. Chern-Simons theory).

The edge excitations for $p + ip$ are described by the Ising chiral conformal field theory.

For the Moore-Read quantum Hall state, the edge theory is a chiral Ising field theory and a free chiral boson.

The chiral Ising edge has three kinds of excitations, labeled by the identity I (no excitation), the edge fermion ψ, and the spin field σ.

The correspondence with the bulk excitations is not a coincidence.

Deser, Jackiw and Templeton; Witten

The meaning of the fusion rule in chiral CFT is that the operator product expansion of two σ fields contains both the identity field I and the fermion ψ.

Our result: the topological entanglement entropy of the full theory can be encoded in the thermodynamic entropy of the edge modes.

Let's consider the interplay between edge and bulk.

The edge is a circle of circumference R. Which bulk quasiparticles are present determines the boundary conditions on the edge modes around the circle.

When there are no bulk quasiparticles, the edge modes in $p + ip$ are those of a free 1+1 dimensional chiral fermion with antiperiodic boundary conditions around the circle.

When there is a single σ quasiparticle in the bulk (i.e. a single vortex in $p + ip$), the boundary conditions change to periodic.

We obtain the entropies from the chiral partition function of the edge modes

$$\chi_0 = \text{tr } e^{-\beta\mathcal{H}}$$

where $\mathcal{H}$ is the Hamiltonian of the edge modes, and β the inverse temperature.

In the corresponding conformal field theory, this is called a character, and is a standard computation.

For no bulk quasiparticles in $p + ip$, we label the (Ising) character as χ_0.
For a single σ quasiparticle (i.e. a vortex), we label it χ_σ.

In general, let χ_a be the chiral CFT character for primary field a. When the bulk has n_a particles of each species a, the holographic partition function is defined as

$$Z = \sum_a n_a \chi_a$$

The 1+1d system here is a purely chiral CFT with space a circle.

This is reminiscent of Cardy's approach to boundary CFT. However, since the 1+1d system is an edge, it can have no boundary!

To extract the entropies from the holographic partition function, we take the limit of a large disc $R/\beta \to \infty$ before we take the temperature to zero.

The interesting piece $\mathcal{S}$ is subleading:

$$\ln(Z) = fR/\beta + \mathcal{S}$$

We showed by direct computation for a general rational conformal field theory that $\mathcal{S}$ is the topological entanglement entropy!

Finding Z and $\mathcal{S}$ is purely an edge computation. The only input from the bulk are the boundary conditions, which in turn depends only on the number of each type of quasiparticle present.

The topological entanglement entropy has two kinds of contributions:

$$\mathcal{S} = S_{\text{bulk}} + S_{\text{edge}}$$

Both are independent of the size of the system: they are subleading terms in the full entropy.

The “bulk” part arises from the degeneracies of the 2+1d bulk quasiparticles.

For two σ quasiparticles, $S_{\text{bulk}} = \ln(2)$. For n of them, $S_{\text{bulk}} = (n/2) \ln 2$.

In general, for n_a bulk quasiparticles of type a

$$S_{\text{bulk}} = \sum_a n_a d_a$$

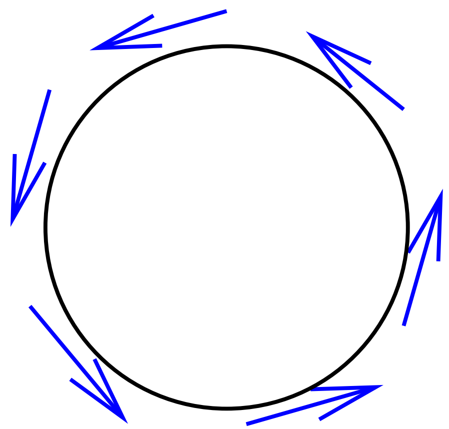
where d_a is the quantum dimension, which is not necessarily an integer!

The “edge” part arises from the 1+1d edge modes. As before,

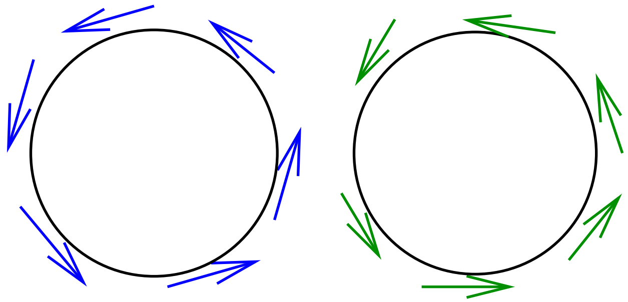
$$\mathcal{S}_{\text{edge}} = -\ln \mathcal{D}$$

This partition function is very useful for understanding the effect of a point contact.

Without the point contact:



After the point contact (a relevant perturbation):

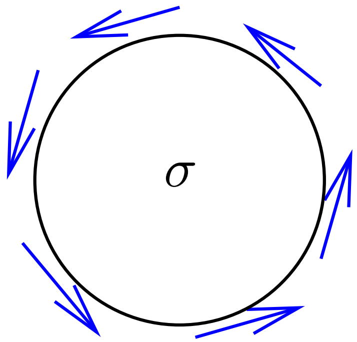


$\mathcal{S}$ is simply the g-function of Affleck and Ludwig!

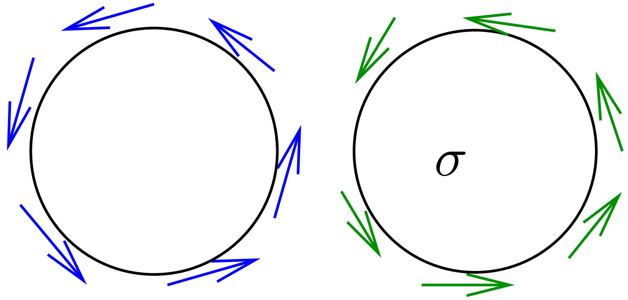
In this context, $\mathcal{S}_{\text{edge}} + \mathcal{S}_{\text{bulk}}$ is often referred to as the “ boundary” entropy. The point contact is the boundary of a boundary!

The entropy lost in the flow is the entanglement entropy of the edge modes on the two halves.

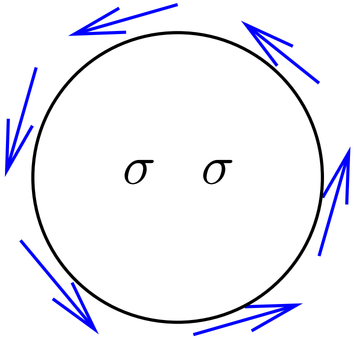
This still works when we include bulk quasiparticles.



$$\mathcal{S}_{UV} = \ln(d_\sigma) - \ln(\mathcal{D}) = \ln(\sqrt{2}) - \ln(2)$$

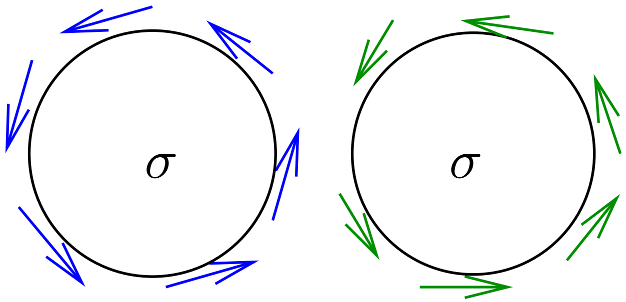


$$\mathcal{S}_{IR} = \ln(d_\sigma) - 2\ln(\mathcal{D}) = \ln(\sqrt{2}) - 2\ln(2) = \mathcal{S}_{UV} - \ln(2)$$



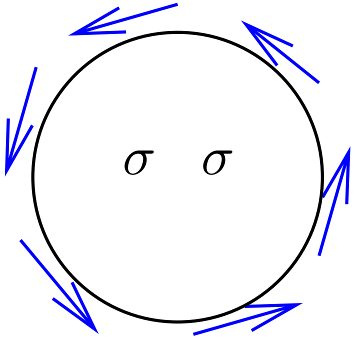
Because $\sigma \cdot \sigma = I + \psi$, there are two channels possible. The “bulk” entanglement entropy is $\ln(2)$, so

$$\mathcal{S}_{UV} = \ln(2) - \ln(\mathcal{D}) = 0$$



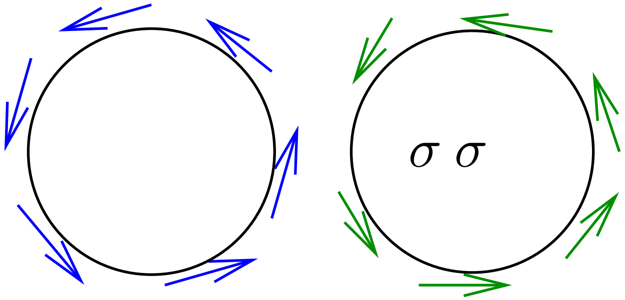
$$\mathcal{S}_{IR} = 2\ln(d_\sigma) - 2\ln(\mathcal{D}) = 2\ln(\sqrt{2}) - 2\ln(2) = \mathcal{S}_{UV} - \ln(2)$$

But what if the bulk quasiparticles are in different places?



The UV is identical:

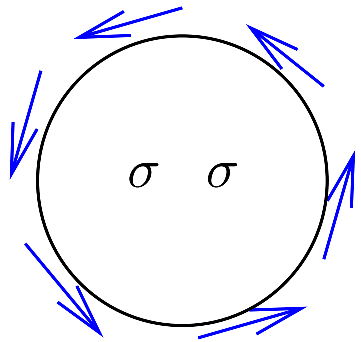
$$\mathcal{S}_{UV} = \ln(2) - \ln(\mathcal{D}) = 0$$



$$\mathcal{S}_{IR} = \ln(2) - 2\ln(\mathcal{D}) = -\ln(2) = \mathcal{S}_{UV} - \ln(2)$$

The same change in entropy!

At the level of the partition functions:

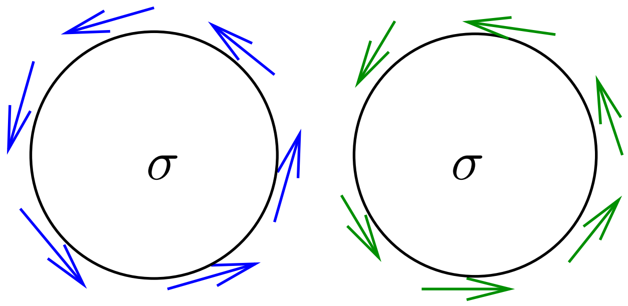


Because $\sigma \cdot \sigma = I + \psi$,

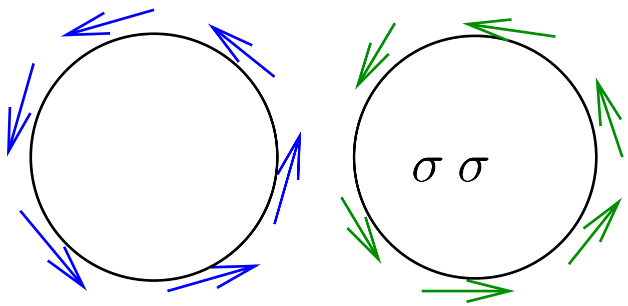
$$Z_{UV} = \chi_I + \chi_\psi$$

Taking the disk large gives the correct $\mathcal{S}_{UV} = 0$.

In the IR, the two halves are independent, so the partition function is the product of the two individual ones.



$$Z_{IR} = (\chi_\sigma)^2$$



$$Z_{IR} = \chi_I(\chi_I + \chi_\psi)$$

To prove this result in general, use conformal field theory.

A modular transformation involves changing the roles of space and Euclidean time, i.e. the radius R of the disk with the inverse temperature β.

The modular S matrix (not to be confused with the Kondo spin $\vec{S}$ or the entropy $\mathcal{S}$) relates the chiral partition functions:

$$\chi_a(R, \beta) = \sum_b S_a^b \chi_b(\beta, R)$$

The fusion coefficients are the integers which encode the fusion rules, e.g. $N_{\sigma\sigma}^{I} = N_{\sigma\sigma}^{\psi} = 1$. The Verlinde formula relates them to S for any rational conformal field theory

$$N_{ab}^{c} = \sum_{j} \frac{S_a^j S_b^j S_j^c}{S_0^j}$$

The original proof is by Moore and Seiberg; Cardy gives a nice proof using boundary conformal field theory.

Define the matrix Q_a, whose elements are

$$(Q_a)^b_c \equiv N^c_{ab}$$

Think of Q as the matrix which fuses one quasiparticle a with a particle c to get quasiparticles b.

The quantum dimension d_a is the largest eigenvalue of Q_a.

Using the Verlinde formula gives

$$d_a = \frac{S^0_a}{S^0_0}$$

and by the unitarity of the modular S matrix

$$\mathcal{D} = 1/S^0_0$$

Cardy, Affleck and Ludwig

In general, the degeneracy of channel c when there are n_a of the ath type of quasiparticle is the $c, 0$ entry of the matrix T

$$T = \prod_a (Q_a)^{n_a}$$

The channel c has quantum dimension d_c.

The bulk entropy in general is therefore

$$\mathcal{S}_{\text{bulk}} = \ln(\sum_c T_{0c} d_c)$$

one finds exactly

$$\begin{aligned} \mathcal{S} &= -\ln \mathcal{D} + \ln(\sum_c T_{0c} d_c) \\ &= \mathcal{S}_{\text{edge}} + \mathcal{S}_{\text{bulk}} \end{aligned}$$

This is holographic: the (subleading) piece of the 2+1d bulk entropy is given entirely in terms of 1+1d CFT character!

For the Moore-Read state, we must include the chiral charge boson φ_c. There are now six quasiparticles:

- I (identity/electron)
- $\sigma e^{\pm i\varphi/(2\sqrt{2})}$ (charge $\pm 1/4$)
- ψ (neutral fermion)
- $e^{\pm i\varphi/\sqrt{2}}$

These are the six primary fields of the (Neveu-Schwarz sector of) the $\mathcal{N} = 2$ superconformal algebra with $c = 3/2$. Milovanovic and Read

These have quantum dimensions

$$1, \sqrt{2}, \sqrt{2}, 1, 1, 1$$

giving

$$\mathcal{D}_{MR} = 2\sqrt{2}$$

Changing the radius of the charge boson gives different Moore-Read states. This of course changes the total quantum dimension. For the $SU(2)$-invariant bosonic Moore-Read state,

$$\mathcal{D}_{SU(2)_2} = 2$$

Lots of details and more results in our papers.

The entire flow due to a point contact can be mapped

- onto the anisotropic Kondo problem ($p + ip$ superconductor)
- onto a variant of the two-channel Kondo problem (Moore-Read)

The connection between 2+1-dimensional topological states and 1+1 dimensional conformal field theory and integrability is a deep one!