Entanglement entropy in extended quantum systems

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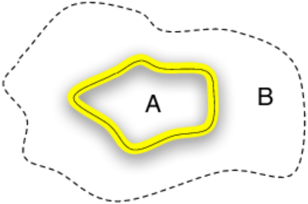
Joint work with John Cardy

hep-th/0405152, cond-mat/0503393, quant-ph/0505193
see also cond-mat/0512586 and unpublished

Entanglement Entropy: what is it?

Quantum system in the ground state $|\Psi\rangle$
The density matrix is $\rho = |\Psi\rangle\langle\Psi|$
$(\mathrm{Tr}\rho^n = 1)$



Alice measures a subset A, Bob the remainder B:
Reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$ $(\rho_B = \mathrm{Tr}_A \rho)$
Entanglement Entropy $\equiv$ Von Neumann entropy of ρ_A:

$$S_A = -\mathrm{Tr}\, \rho_A \ln \rho_A$$

Note: $S_A = S_B$ if ρ corresponds to a pure state

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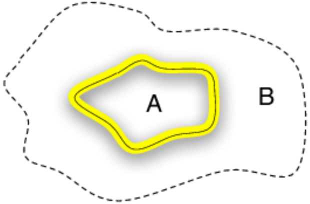

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What is the meaning of S_A?

It is the amount of information that A and B are shearing
The amount of quantum correlations between A and B

Area law and criticality

- Area Law: $S_A \propto \mathcal{A}$ [Non extensive]

Srednicki '93

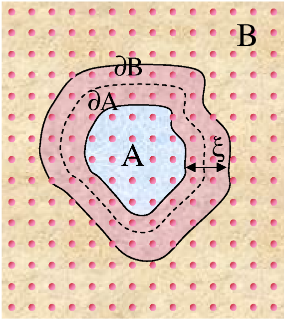
↓

(lots of works)

↓

Wolf et al '07

Only in gapped systems



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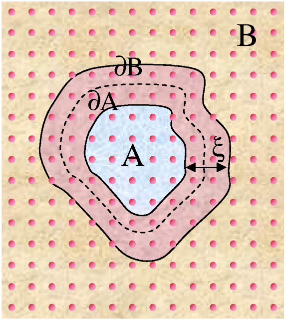

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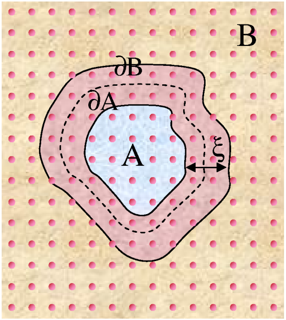

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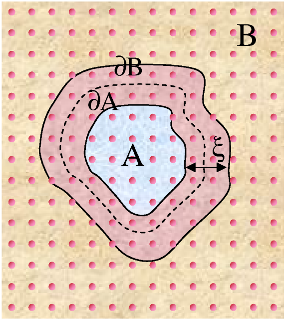

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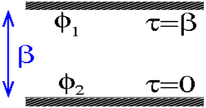
- Extensive review by Amico et al. [quant-ph/0703044]

Entanglement Entropy and path integral

Lattice QFT in 1+1 dimensions: $\{\hat{\phi}(x)\}$ a set of fundamental fields with eigenvalues $\{\phi(x)\}$ and eigenstates $\otimes_x|\{\phi(x)\}\rangle$
The density matrix at temperature β^{-1} is $(Z = \mathrm{Tr}\, e^{-\beta\hat{H}})$

$$\rho(\{\phi_1(x)\}|\{\phi_2(x)\}) = Z^{-1}\langle\{\phi_2(x)\}|e^{-\beta\hat{H}}|\{\phi_1(x)\}\rangle$$

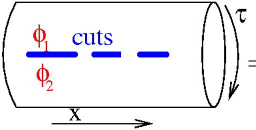
Euclidean path integral:

$\rho =$ 

$$= \int \frac{[d\phi(x,\tau)]}{Z} \prod_x \delta(\phi(x,0)-\phi_2(x)) \prod_x \delta(\phi(x,\beta)-\phi_1(x))\, e^{-S_E}$$

$S_E = \int_0^\beta L_E\, d\tau$, with L_E the Euclidean Lagrangian

The trace sews together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β.
$A = (u_1, v_1), \ldots, (u_N, v_N)$: ρ_A sewing together only those points x which are not in A, leaving open cuts for (u_j, v_j) along the the line $\tau = 0$.

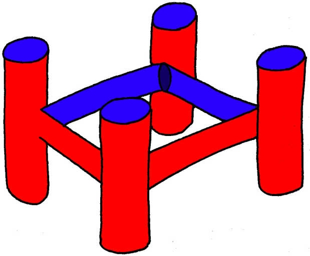
$\rho_A =$ 

$$= \int_{x\in B} [d\phi(x,0)]\delta(\phi(x,\beta) - \phi(x,0))\rho$$

Replica trick

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n$$

$\mathrm{Tr}\rho_A^n$ (for integer n) is the partition function on n of the above cylinders attached to form an $n-$sheeted Riemann surface



$$= \text{“}\rho_A^{ij}\rho_A^{jk}\rho_A^{kl}\rho_A^{li}\text{”}$$

$\mathrm{Tr}\rho_A^n$ has a unique analytic continuation to $\mathrm{Re}\, n > 1$ and that its first derivative at $n = 1$ gives the required entropy:

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$$

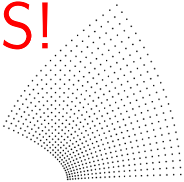
CFT: a remind

- A physical systems at a quantum critical point is scale invariant
  $\langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle = b^{2\Delta_\phi}\langle \phi(b\mathbf{r}_1)\phi(b\mathbf{r}_2)\rangle \qquad \langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle = |\mathbf{r}_1 - \mathbf{r}_2|^{-2\Delta_\phi}$
- A Hamiltonian that is invariant under translations, rotations, and scaling transformations has usually the symmetry of the larger *conformal* group defined as the set of transformations that do not change the angles.
- In 2D the consequences are extraordinary:

YOU DON'T NEED THIS!

all the analytic functions $w(z)$ are conformal

$$\langle \phi(z_1)\phi(z_2)\rangle = |w'(z_1)w'(z_2)|^{2\Delta_\phi}\langle \phi(w(z_1))\phi(w(z_2))\rangle$$



- Under an arbitrary transformation $x^\mu \to x^\mu + \epsilon^\mu$

$$S \to S + \delta S, \qquad \text{with} \quad \delta S = \int d^2x T^{\mu\nu}\partial_\mu \epsilon_\nu$$

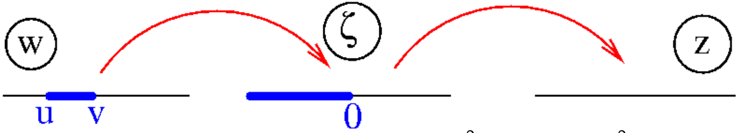
Under a conformal transformation $w \to z$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\frac{z'''z' - 3/2 z''^2}{z'^2}$$

Entropy and CFT

Single interval (u, v). We need $Z_n/Z^n = \langle 0|0\rangle_{\mathcal{R}_n} \Rightarrow$ compute $\langle T(w)\rangle_{\mathcal{R}_n}$

$$w \to \zeta = \frac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow w \to z = \left(\frac{w-u}{w-v}\right)^{1/n}$$



$$\langle T(z)\rangle_{\mathbf{C}} = 0 \Rightarrow \qquad \langle T(w)\rangle_{\mathcal{R}_n} = \frac{c(1-(1/n)^2)}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}$$

To be compared with the Conformal Ward identities

$$\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}}}{\langle\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}}} = \frac{\Delta_\Phi (v-u)^2}{(w-u)^2(w-v)^2}$$

Z_n/Z^n transforms under conformal transformations as n^{th} power of the two point function of a (fake) primary field on the plane with scaling dimension

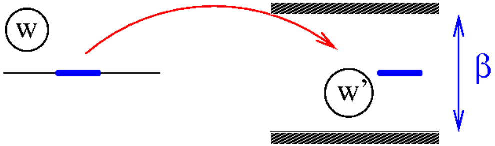
$$\Delta_\Phi = \overline{\Delta}_\Phi = \frac{c}{24}\left(1-\frac{1}{n^2}\right) \Rightarrow \qquad \mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z^n} = c_n\left(\frac{v-u}{a}\right)^{-(c/6)(n-1/n)}$$

Finally with the replica trick $(v - u = \ell)$

$$S_A = \frac{c}{3}\ln\frac{\ell}{a} + c_1'$$

Generalization I

- Finite temperature: map the plane into a cylinder



$$w \to w' = \frac{\beta}{2\pi} \ln w$$

$$S_A = \frac{c}{3} \log\left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta}\right) + c_1' \simeq \begin{cases} \dfrac{\pi c}{3} \dfrac{\ell}{\beta}\,, & \ell \gg \beta \qquad \text{classical extensive} \\ \dfrac{c}{3} \log \dfrac{\ell}{a}\,, & \ell \ll \beta \qquad T = 0 \text{ non} - \text{extensive} \end{cases}$$

see also [Korepin '04]

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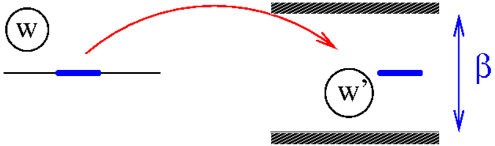

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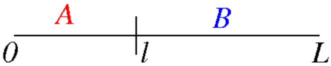
- Finite size: orient the branch cut perpendicular to the axis
  $\beta \to L$ and $w \to iw$

$$S_A = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c_1'$$

It is symmetric under $\ell \to L - \ell$. It is maximal when $\ell = L/2$

Generalization II

Open boundaries: semi-infinite system



If $L=\infty$ and $T=0$, it is uniformised by $z=\left(\frac{w-i\ell}{w+i\ell}\right)^{1/n}$

$$\mathrm{Tr}\,\rho_A^n \simeq \tilde{c}_n \left(\frac{2\ell}{a}\right)^{(c/12)(n-1/n)} \Rightarrow S_A \simeq \frac{c}{6}\log\frac{2\ell}{a} + \tilde{c}'_1$$

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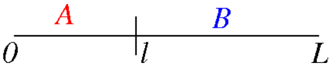

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at finite temperature β^{-1}

$$S_A(\beta) \simeq \frac{c}{6}\log\left(\frac{\beta}{\pi a}\sinh\frac{2\pi\ell}{\beta}\right)+\tilde{c}_1'$$

and finite size

$$S_A(L) \simeq \frac{c}{6}\log\left(\frac{2L}{\pi a}\sin\frac{\pi\ell}{L}\right)+\tilde{c}_1'$$

$\tilde{c}_1' - c_1'/2 = \ln g$ boundary entropy
[Affleck, Ludwig]

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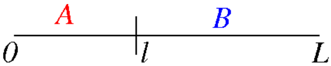

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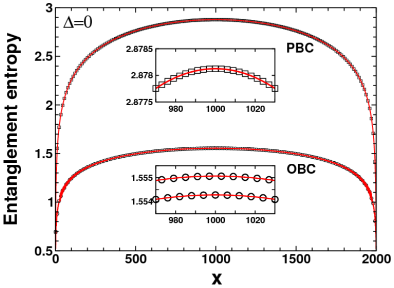
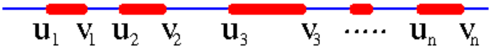


[From Laflorencie et al '06]

Generalization III

- General case



$$S_A = \frac{c}{3}\left(\sum_{j\leq k}\log\frac{v_k - u_j}{a} - \sum_{j<k}\log\frac{u_k - u_j}{a} - \sum_{j<k}\log\frac{v_k - v_j}{a}\right) + Nc_1'$$

A similar expression holds in the case of a boundary, with half of the w_i corresponding to the image points

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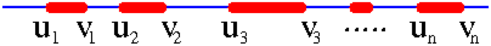

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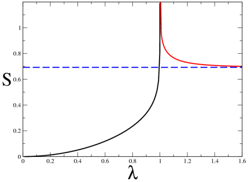
- Non-critical systems

Following the line of the c-theorem proof we showed

$$S_A = \mathcal{A}\frac{c}{6}\log\frac{\xi}{a}$$

$\mathcal{A}$ is the number of boundary points between A and B (1D area).

EG: Ising model in a transverse magnetic field with corner transfer matrix



For $\lambda \to 1$

$S \simeq \frac{1}{12}\log\xi$

How does entanglement evolve from a state that is *not* an eigenstate?

Example

Ising model in a transverse field with $H(h)$:

- Prepare the system in a pure state $|\psi_0\rangle$ (ground state of $H(h_0)$)
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How can we study this problem with QFT?

$$\langle\psi''(x'')|\rho(t)|\psi'(x')\rangle = Z_1^{-1}\langle\psi''(x'')|e^{-itH} \quad |\psi_0(x)\rangle\langle\psi_0(x)|e^{+itH} \quad |\psi'(x')\rangle$$

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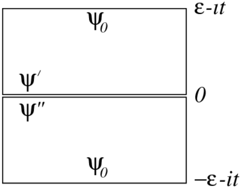

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Dynamics of Entanglement [John Cardy talk]

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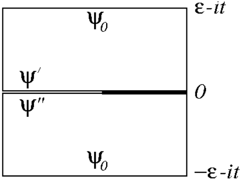

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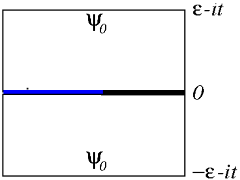

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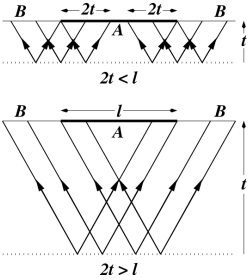
With the "usual" mapping and for $t, \ell \gg \tau_0$ $\quad S_A(t) \simeq \begin{cases} \dfrac{\pi c t}{6\tau_0} & t < \ell/2 \\[2ex] \dfrac{\pi c\, \ell}{12\tau_0} & t > \ell/2 \end{cases}$

$S_A(t)$ increases linearly until it saturates at $t = \ell/2$: horizon effect



S_A is proportional to the number of particles that emitted from a region of size τ_0 reach one A and the other B.

CFT result and horizon

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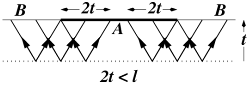
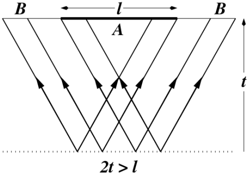


S_A is proportional to the number of particles that emitted from a region of size τ_0 reach one A and the other B.

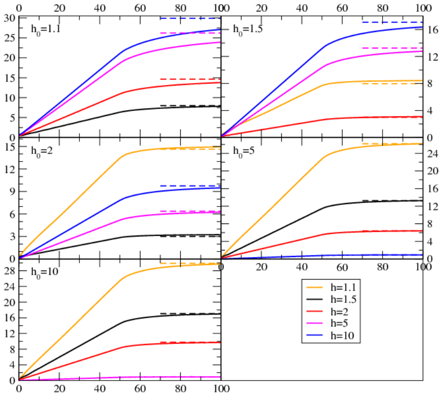


Generalizable to the case when A consists of several disjoint intervals. $S_A(t)$ is not always non-decreasing:

EG, $A =$ regular array of intervals S_A oscillates in a saw-tooth fashion

Lattice calculation

Transverse Ising chain $H_I(h) = -\frac{1}{2}\sum_j [\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z]$



Crossover always at $t^* = \ell/2$!!

$S_\ell(t)$ increases linearly with time up to $t^* = \ell/2$, but it does not saturated immediately:

There are particles moving slowly as a consequence of $v_p = dE_p/dp$

Local Quench [unpublished]

We physically cut a spin chain into two A and B parts

At time $t = 0$ we join the two parts

How does the entanglement entropy evolve?

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$$S_{A'} \simeq \begin{cases} \dfrac{c}{3}\ln t + \dfrac{c}{6}\ln \ell + \dfrac{c}{6}\ln \dfrac{\ell - t}{\ell + t} & t < \ell \\[2ex] \dfrac{c}{3}\ln \ell & t > \ell \end{cases}$$

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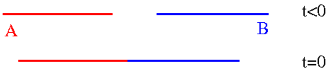

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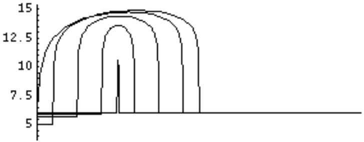
- If the slit has the joining point inside:



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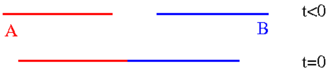

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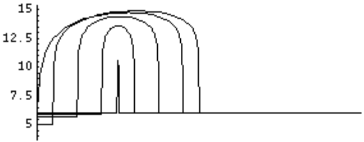

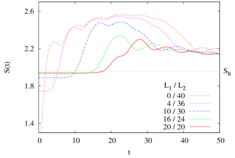

- Everything agrees with numerics by Eisler and Peschel '07