Pairwise entanglement in 1d spin chains

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Superconductivity
Mesoscopics
Theory group

Materials and Technologies
for Information and communication Sciences

ERA-Pilot Roadmap :
Quantum Information Sciences & Technologies :

Outline

- **General ideas.**

- **Part I**: Entanglement & QPT

  Summary at T=0

  In collaboration with R. Fazio (Pisa), A. Osterloh (Hannover).

  Thermal Entanglement close to QPT

  In collaboration with D. Patanè (Catania).

- **Part II: Entanglement & separable states in low dimensional systems**

  In collaboration with D. Patanè (Catania), F. Baroni, A. Fubini, V. Tognetti, P. Verrucchi (Firenze)

- **Part III:** Bound entanglement in spin chains **(with D. Patane' and R. Fazio).**

General Ideas: Entangled Vs Not entangled states.

Example: 2 spins (or qubits)

- **Pure states:** $|\Psi\rangle$

Separable: $|\uparrow\rangle \otimes |\downarrow\rangle = |\uparrow\downarrow\rangle$

$$(|\uparrow\rangle + |\downarrow\rangle) \otimes |\uparrow\rangle = |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle$$

Entangled: $|\Phi\rangle = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$

- **Mixed states:** $\sum_i p_i |\psi_i\rangle\langle\psi_i|$

Separable: $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$

Entangled: $(1-p)|\Phi\rangle\langle\Phi| + p|\uparrow\uparrow\rangle\langle\uparrow\uparrow|$ $\quad 0 \leq p < 1$

Classical Vs quantum correlations

- A separable state (not entangled) may contain classical correlation:

$$\begin{aligned}\rho &= \sum_i p_i(|A,B\rangle\langle B,A|)_i \\ &= \tfrac{2}{4}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \tfrac{1}{4}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + \tfrac{1}{4}|\uparrow\uparrow\rangle\langle\uparrow\uparrow|\end{aligned}$$

Measure of B $B = |\downarrow\rangle$

$$\rho = \frac{2}{3}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \frac{1}{3}|\uparrow\downarrow\rangle\langle\uparrow\downarrow|$$

Once $B = |\downarrow\rangle$, the probability to find $A = |\downarrow\rangle$ is double than the probabilty to find $A = |\uparrow\rangle$: **A and B are classically correlated.**

General aim:

Quantify Entanglement in many body systems.

Possible questions:

- Entanglement as a resource (q-compuation....)
- Correlation Vs Entanglement
- Entanglement and Critical phenomena ?

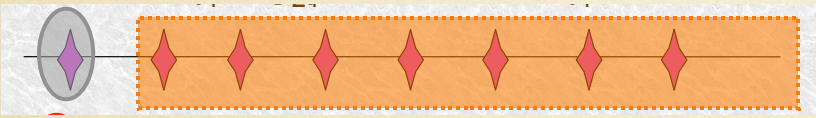
Entanglement measures

Review: Horodecki's family quant-ph/0702225

Rationale: how many Bell states are 'contained ' in a given state? (Bennet et al., PRA 1996.)

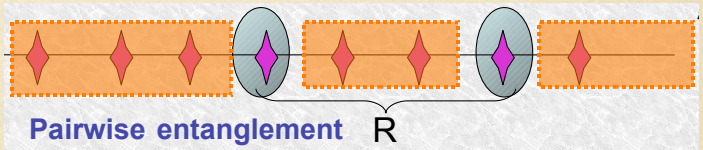
- **Bipartite entanglement for pure qubit states:** One-tangle

  von Neumann Entropy: $E = Tr\rho_1 \log_2 \rho_1 \quad \leftrightarrow \quad 4det\rho_1$



Coffman, Kundu,Wootters 2000.
Osterloh, Siewert 2004.

- **Bipartite entanglement for mixed states.**



- **For qubits:** Concurrence.

  $C(R) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$

  Wootters 1998.

  λ_α are eigenvalues of $S = \rho_2\sigma_y \otimes \sigma_y\rho_2^*\sigma_y \otimes \sigma_y$

- **For higher dimensional local Hilbert space:**

The Peres criterium $\longleftrightarrow$ **Negativity**: sum of negative eigenvalues of partial transpose density matrix.

Horodecki's family 1996; Peres 1996.

QPT in 1d-Anisotropic XY models

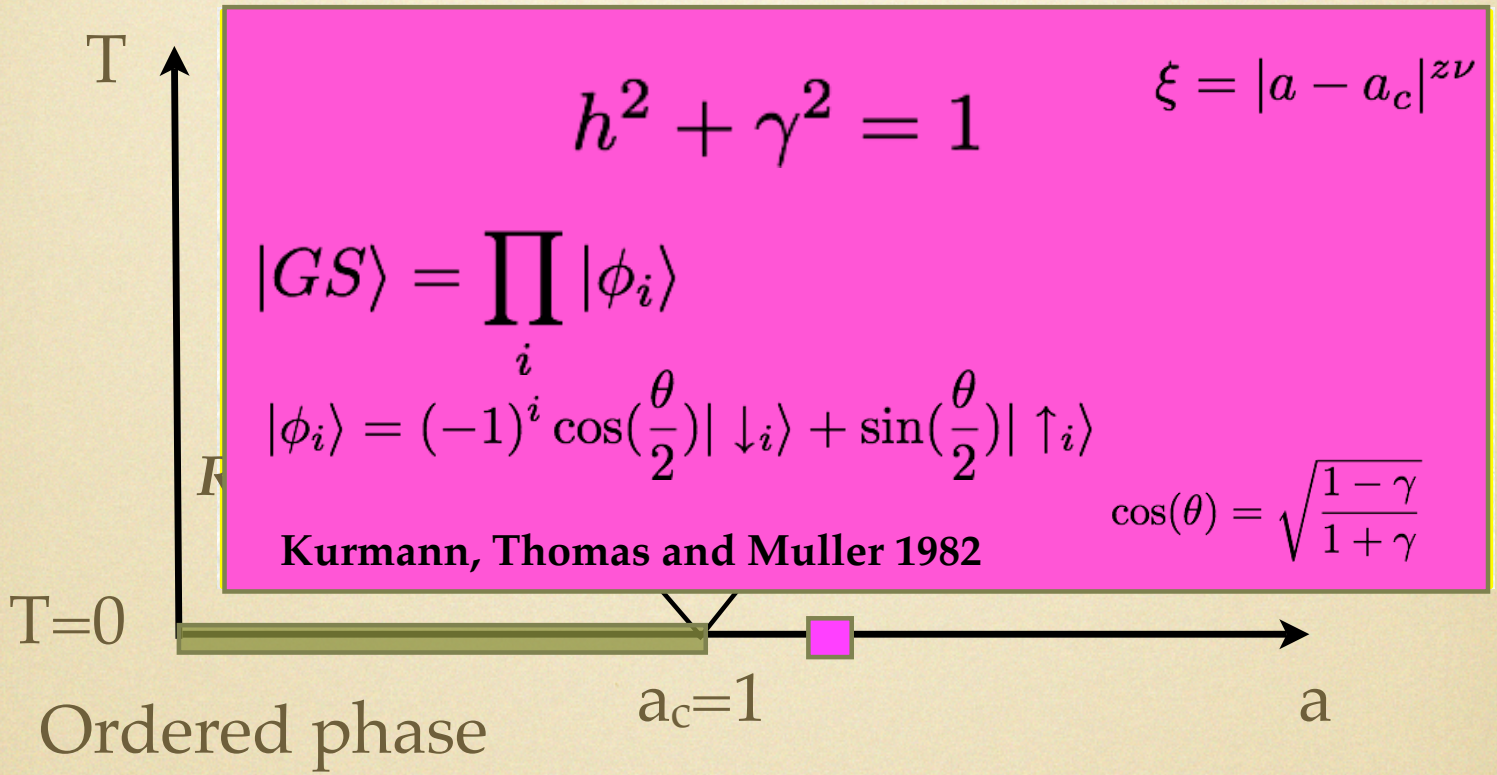
$$H = J \sum_i (1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y - h\sigma_i^z$$

- Completely integrable.
- Quantum phase transition at $a_c = 1$.

$$a = \frac{2h}{J}$$

Lieb, Schulz, Mattis Ann. Phys.NY 16, 407 (1961);
Barouch, McCoy, Dresden PRA 2, 1075 (1970);
Barouch, McCoy PRA 3, 786 (1971);
Pfeuty Ann. Phys.NY 57, 79 (1970).

Cross-over phase diagram for the quantum Ising models



$$h^2 + \gamma^2 = 1$$

$$\xi = |a - a_c|^{z\nu}$$

$$|GS\rangle = \prod_i |\phi_i\rangle$$

$$|\phi_i\rangle = (-1)^i \cos(\frac{\theta}{2})|\downarrow_i\rangle + \sin(\frac{\theta}{2})|\uparrow_i\rangle$$

$$\cos(\theta) = \sqrt{\frac{1-\gamma}{1+\gamma}}$$

Kurmann, Thomas and Muller 1982

Chakravarty, Halperin, Nelson 1989;
Chubukov, Sachdev, Ye (1994);
Sachdev 1996;
Kopp, Chakravarty (2005).

Part I: Entanglement close to QPT

Possible questions:

- Correlation Vs Entanglement ?
- Critical properties ?
- Universality ?

Preskill 2000
Arnesen, Bose, Vedral 2001
Gunlicke, Bose, Kendon, Vedral 2001

Selected results at T=0

- **Pairwise entanglement close to QPT:**

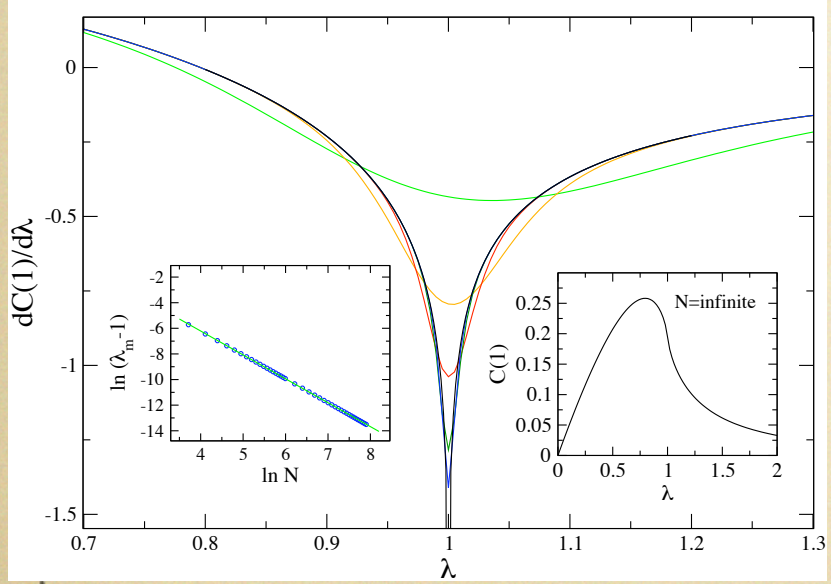
  Critical change of Concurrence at the quantum critical point.

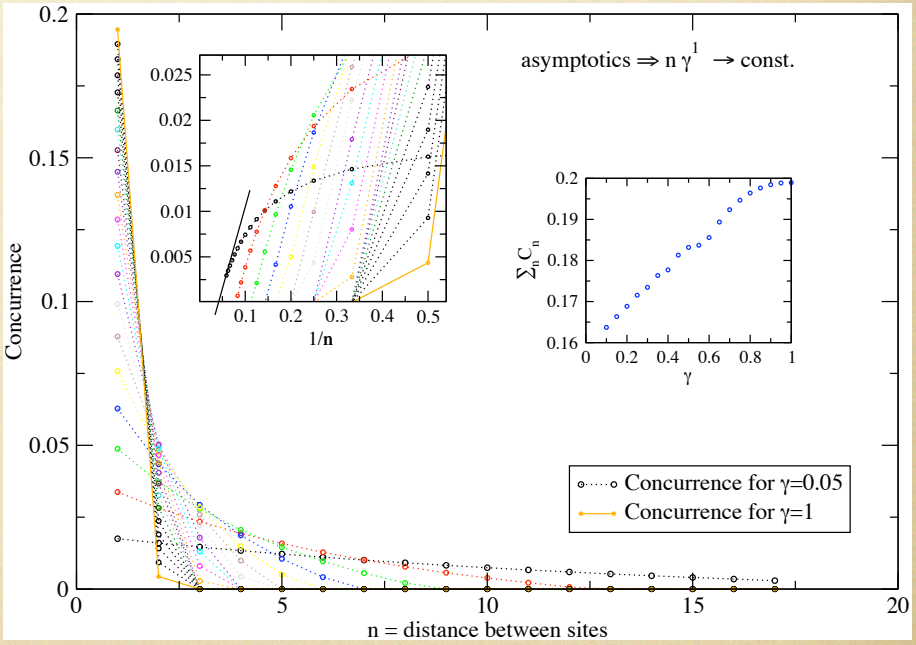
  Finite size scaling.

  At criticality: In general NO long range Concurrence.

  Osterloh, Amico, Falci and Fazio, Nature (2002); Osborne, Nielsen PRA 2002.

Fig.1 Osterloh et al.





$$\partial_a C(1) = \frac{8}{3\pi}|a - a_c|$$

$$R \sim \gamma^{-1}$$

Selected results at T=0

- Second order QPT are marked by singularities in the first derivative of a given entanglement measure; first order QPT are marked by anomalies in the entanglement itself.

  Wu, Sarandy, Lidar Phys. Rev. Lett. 2004; Wu, Sarandy, Lidar, Sham 2005. Yang 2005; Gu, Tian, Lin 2005. See also: Verstrate, Popp, Cirac 2004; Jin, Korepin 2004.

- **Multipartite entanglement:**

  QPT characterized by a maximum of the multipartite/bipartite entanglement ratio.

  Critical Block-entanglement is universal: ~(c/3) logL.

  (Vidal, Latorre, Rico, Kitaev 2003; Calabrese and Cardy 2004-06; Its, Jin, Korepin 2005).

  Localizable Entanglement: 'topological order' (Haldane phase); critical diverging length of the localizable entang. lenght.

  (Popp, Verstraete, Cirac, Martin-Delgado 2004)

$$H = H_0 + \sum_l \lambda_l A_l$$

, The reduced density operator is a function of correlators. Then

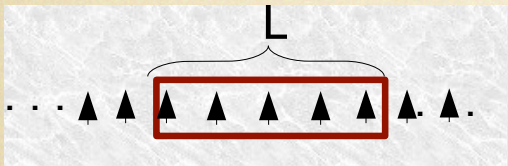
$$M(\langle A_l \rangle) = M\left(\frac{\partial E_{gs}}{\partial \lambda_l}\right)$$

Why multiparticle entanglement?

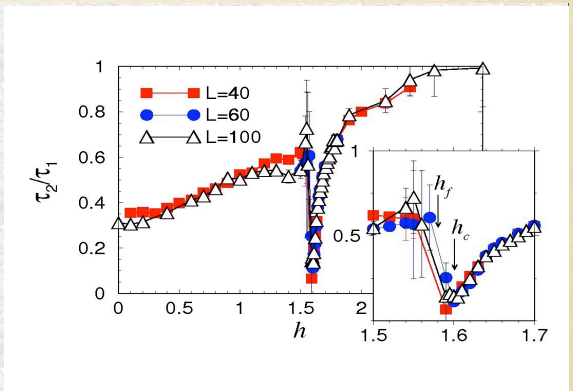
- **Quantum Phase Transition:**

  Enhancement of multiparticle entanglement

  G. Vidal et al. , Phys.Rev.Lett. 90, 227902 (2003); T. Roscilde et al., Phys. Rev. Lett. 93, 167203 (2004).



$$S = \frac{c}{3} \log L$$



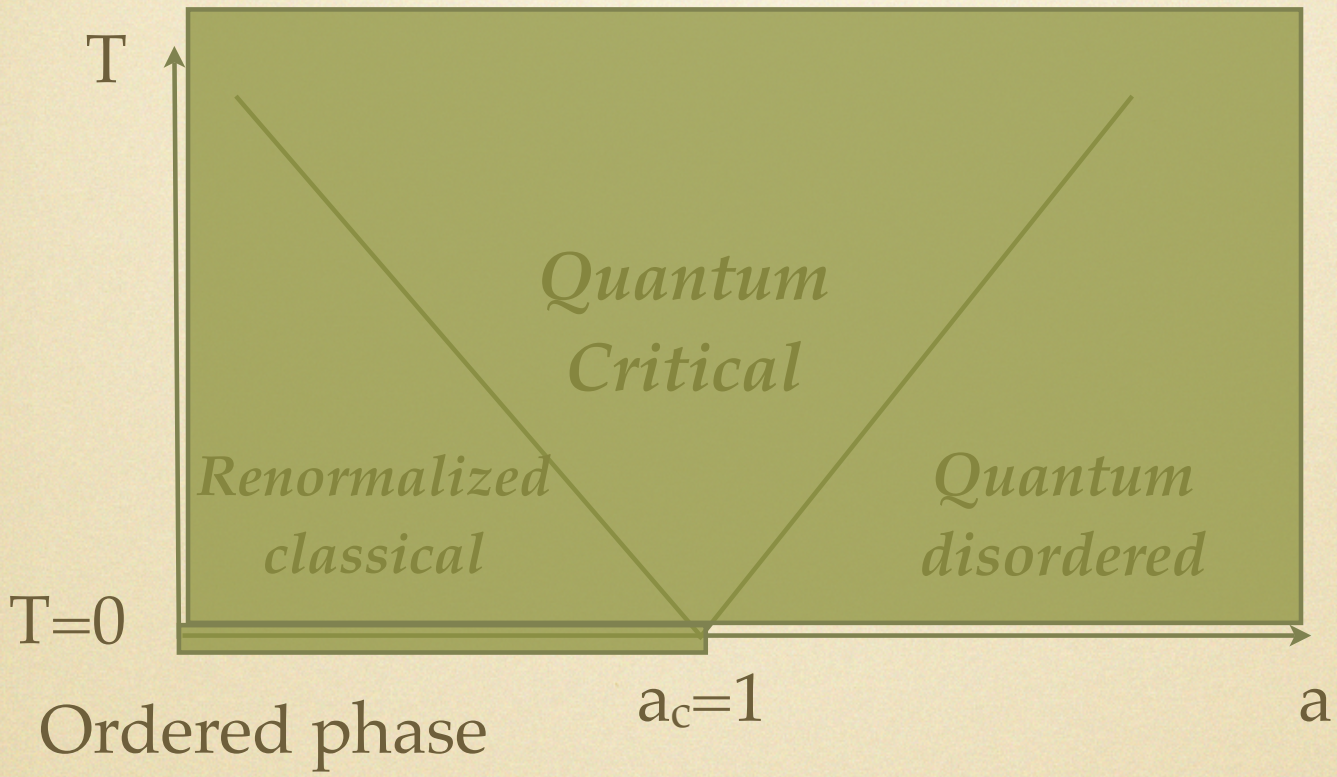
- **Simulations of strongly correlated systems:**

  - Failure DMRG for unbounded block entangl. (i.e. 1D critical systems or D>2).

  - New algorithms taking into account long range entanglement (i.e. PEPS, graph states..)

    G. Vidal Phys. Rev. Lett. 93, 040502 (2004); F. Verstraete and J. I. Cirac; cond-mat/0407066; G. Vidal, cond-mat/0512165; S. Anders, Phys. Rev. Lett. 97, 107206 (2006).

Thermal entanglement close to a QPT



Scaling

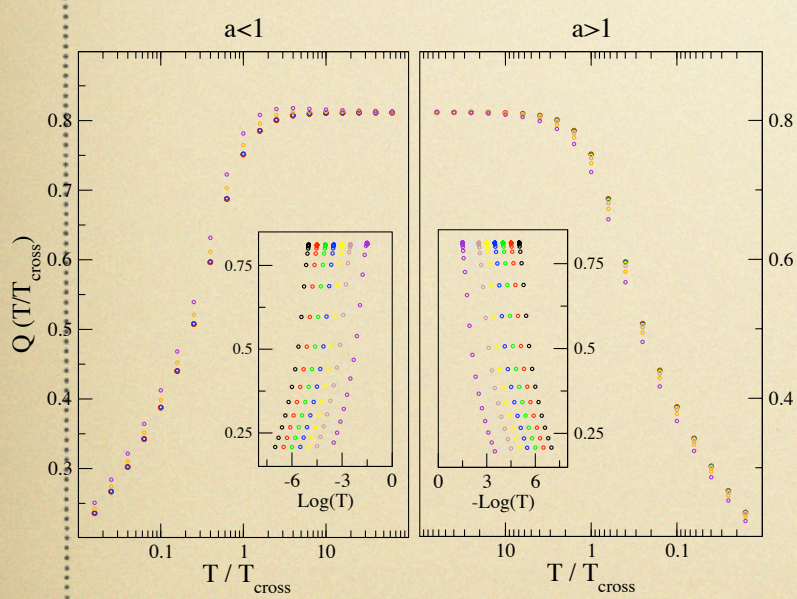
Sensitivity to quantum fluctuations:

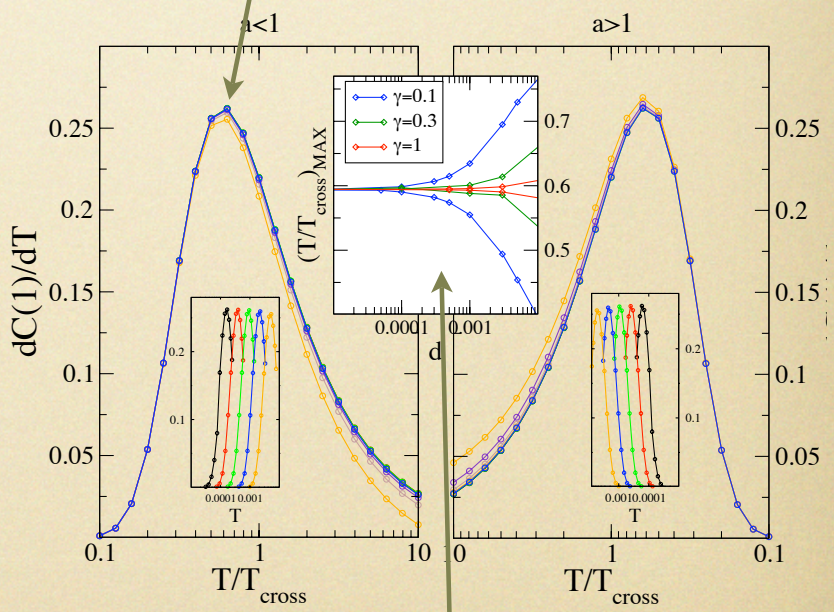
$$\partial_a C(1) \approx \ln\left[T^{\Upsilon} Q\left(\frac{T}{T_{cross}}\right)\right]$$

Sensitivity to thermal fluctuations:

$$\partial_T C(1) \approx P\left(\frac{T}{T_{cross}}\right)$$

$$T^* = \alpha T_{cross} \qquad \alpha \sim 0.595$$





$$T_{cross} = |a - a_c|^{z\nu}$$

Universal T^*

Amico and Patane' 2006

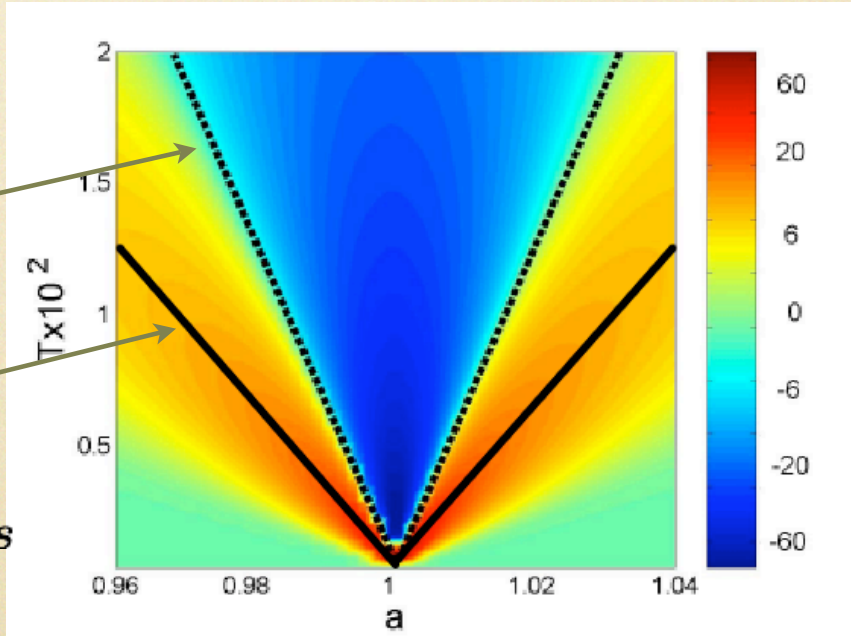
Entanglement crossover

- **How is the entanglement affected by the combinations of thermal and quantum fluctuations?**

$$\partial_T \left[\partial_a C(R)\right]$$

$$T^*$$

$$T_M = 0.290 T_{cross}$$

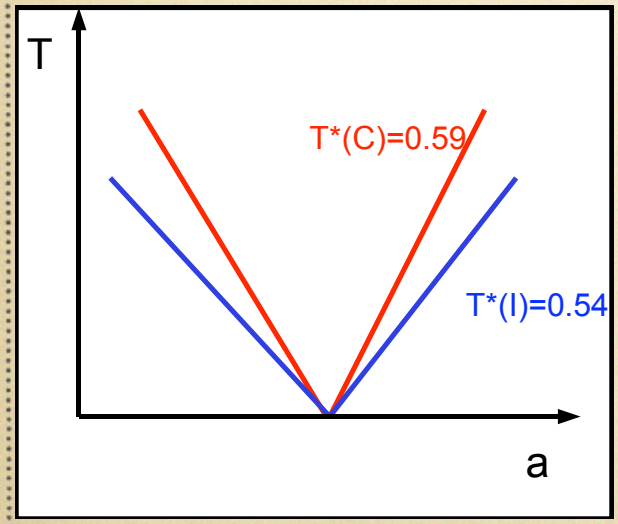


<u>Temperature is a strong effect in the quantum critical region and around T_M</u>

Entanglement Vs Classical correlations

How typical is the behaviour of Concurrence?

Finite Temperature Correlations ⇨ Classical (thermal ensemble) average & Quantum entanglement



Total mutual information:

$$I_{ij} = S_i + S_j - S_{ij}$$

Vedral 2002; Groisman, Popescu, and Winter 2004; Anfossi, Giorda, Montorsi 2005.

Results:

Same "phenomenology" of C_{ij}

Crossover Temp. altered by < 10%

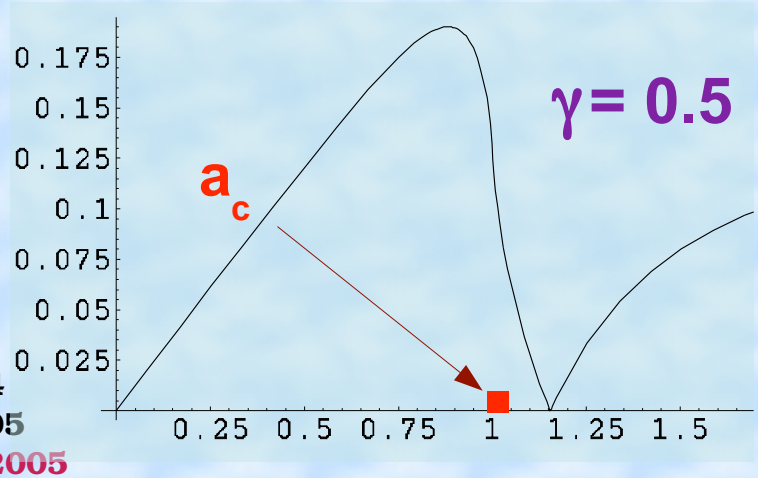
Part II: Factorized ground state

$$\mathcal{H}(j_x, j_y, j_z) = J \sum_i j_x S_i^x S_{i+1}^x + j_y S_i^y S_{i+1}^y + j_z S_i^z S_{i+1}^z - h S_i^z$$

$$h = h_{\mathrm{f}} \equiv \sqrt{(j_x + j_z)(j_y + j_z)}$$

Uncorrelated real space factorized GS

The One tangle is zero



Kurman et al., Physica A 112 (1982)
Roscilde,Verrucchi,Fubini,Haas, Tognetti 2004
Fubini, Roscilde, Tognetti, Tusa, Verrucchi, 2005
2d: Roscilde,Verrucchi,Fubini,Haas, Tognetti 2005

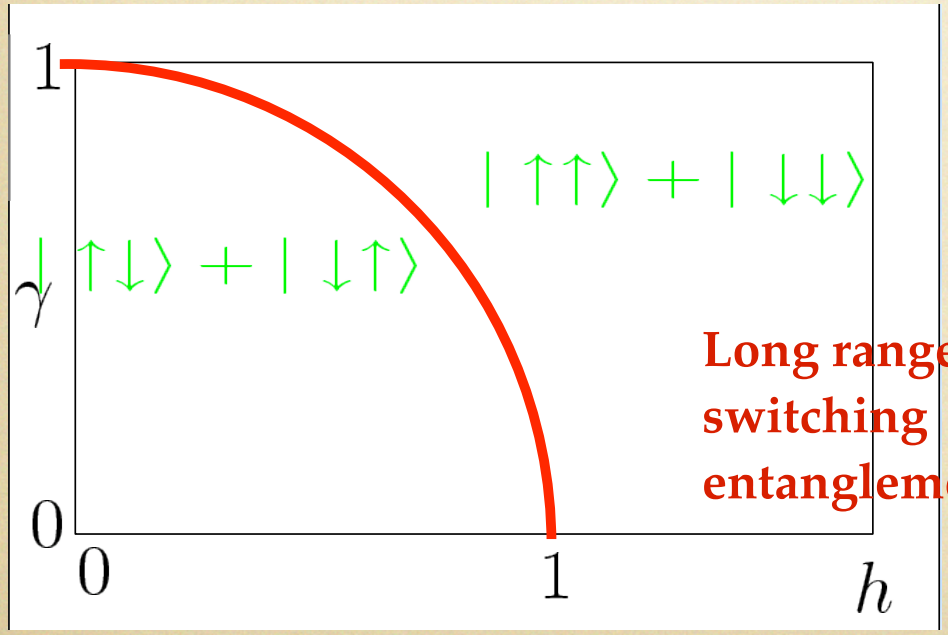
Explanation of Kurman's result

$$C_r = 2\max\{0, C_r^I, C_r^{II}\}$$

$$C_r^I = \left|g_r^{xx} + g_r^{yy}\right| - \sqrt{\left(\frac{1}{4} + g_r^{zz}\right)^2 - M_z^2}$$

$$C_r^{II} = \left|g_r^{xx} - g_r^{yy}\right| + g_r^{zz} - \frac{1}{4}$$

- C^I and C^{II} reflect antiparallel and parallel entanglement respectively.
  (**Fubini, Roscilde, Tognetti, Tusa, Verrucchi 2006**)



Long range reshuffling of the ground states switching from parallel to antiparallel entanglement.

XY model

Divergence of entanglement range at: $h_f = \sqrt{1-\gamma^2}$

$$R^{XY} \propto \left(\ln \frac{1-\gamma}{1+\gamma}\right)^{-1} \ln |h - h_f|^{-1}.$$

Used exact results by Barouch, McCoy 1971.

XXZ model: $h_f = 1 + j_z$

$$R^{XXZ} \propto (h - h_s)^{-\theta/4}$$

$$\theta = 2 + \frac{4\sqrt{h-h_s}}{\pi \tan(\pi\eta/2)\tan(\pi\eta)}$$

$$\eta = \frac{1}{\pi}\arccos(j_z)$$

Used exact results of Lukyanov 1997-98;
Hikihara, Furusaki 2004.
summariezed in Jin, Korepin 2004.

Amico, Baroni, Fubini, Patane', Tognetti, Verrucchi PRA (2006).

Part III: Multipartite entanglement in spin chains

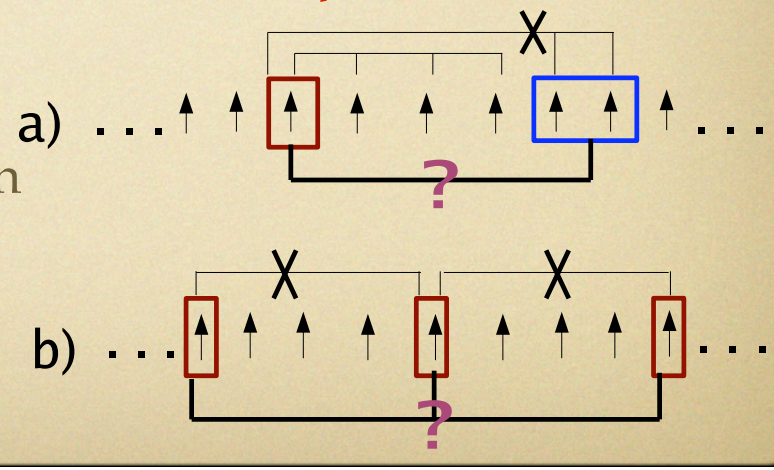
Motivations: analysis of one-tangle and block entropies reveal multipartite entanglement....BUT: ***How is it shared?***

- **Strategy**: Search for configurations/regimes where the two-particle entanglement is known to vanish.

Example for three spin entanglement:

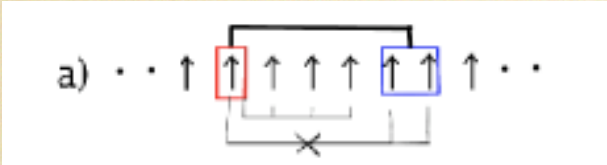
i) If the spin are distant enough then there is ***no two-spin*** entanglement....

ii) Then, pairwise entanglement between ***suitably distant blocks is a measure of genuine multiparticle entanglement.***

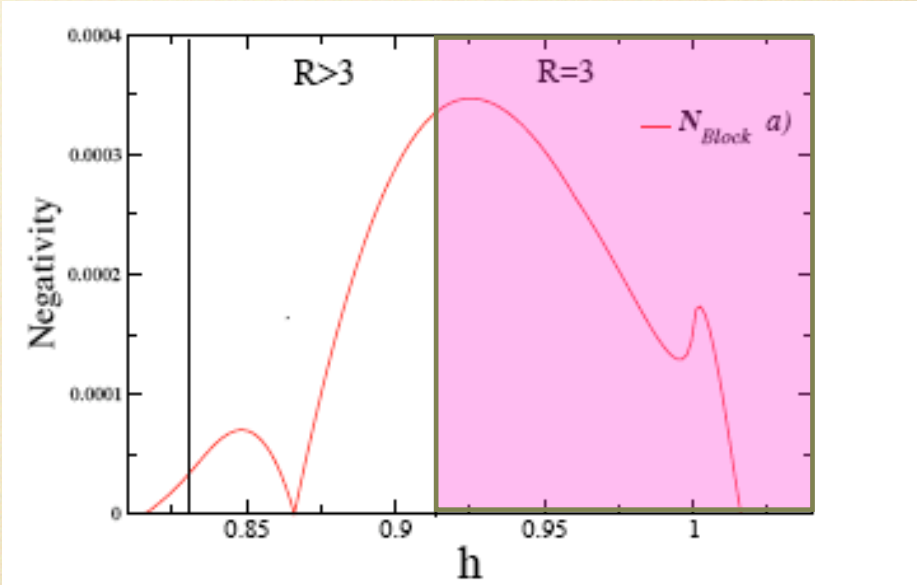


Patane', Fazio, Amico, arXiv:0705.0386

Multi spin entanglement in the ground state of the quantum XY models



γ=0.5



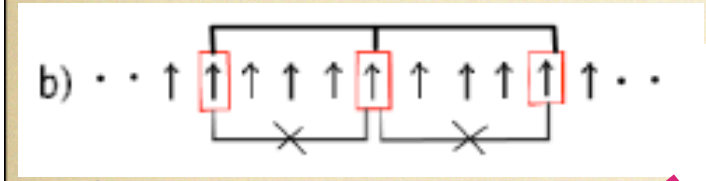
Spin/block entanglement without spin-spin entanglement for distance d=4.

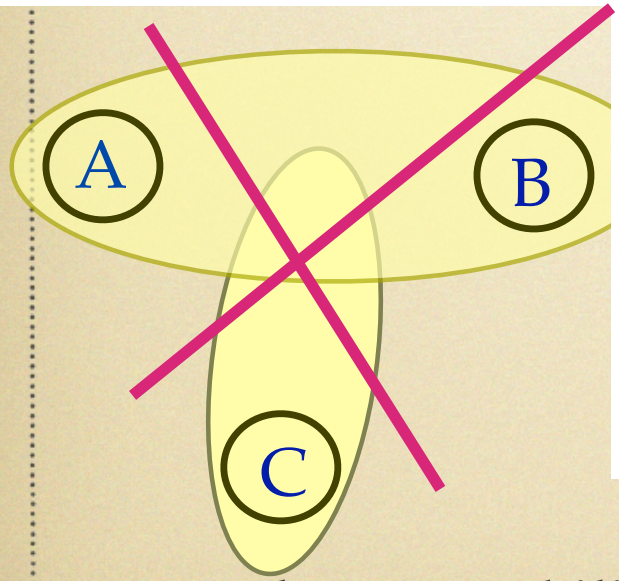
Amico, Fazio, Patanè 2007

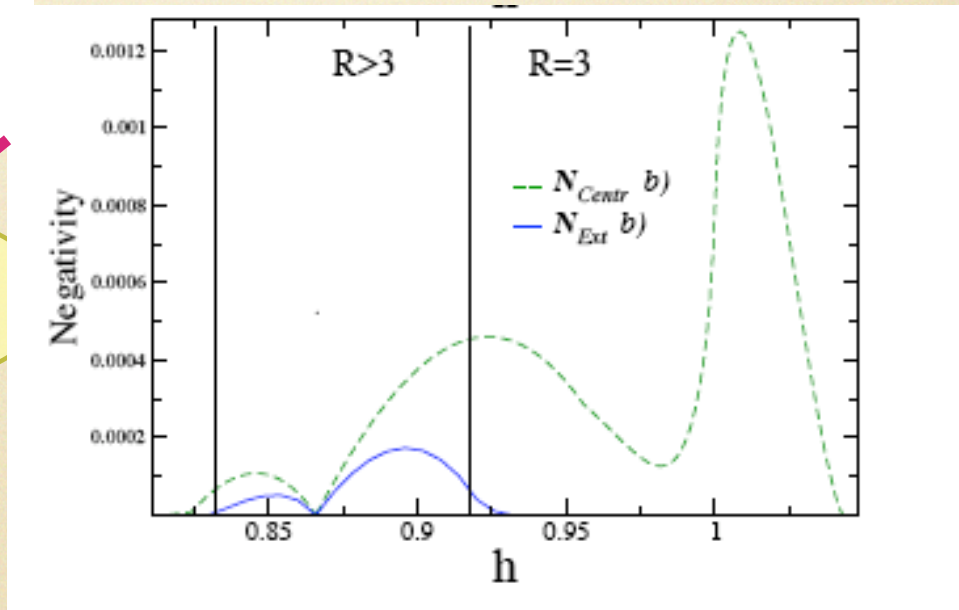
Bound Entanglement in the ground state of the quantum XY model

The ent. is bound: isn't possible to distill maximally entangled (pure) states between parts of the system in a mixed state; neither with local operation nor with classical communications, and not even asymptotically with infinite supply of copies of the state.

It can be useful in certain protocols.







Incomplete separability: Bound Entangled states.

Horodecki et. al PRL 1998; Dur and Cirac PRA 2000; Dur 2001.

Patane', Fazio, Amico, arXiv:0705.0386

Bound entanglement in quantum spin chains at finite temperature

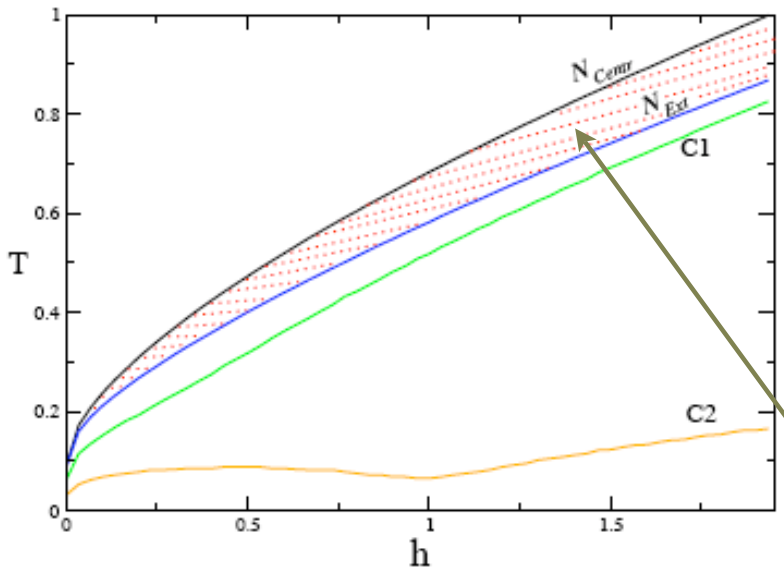
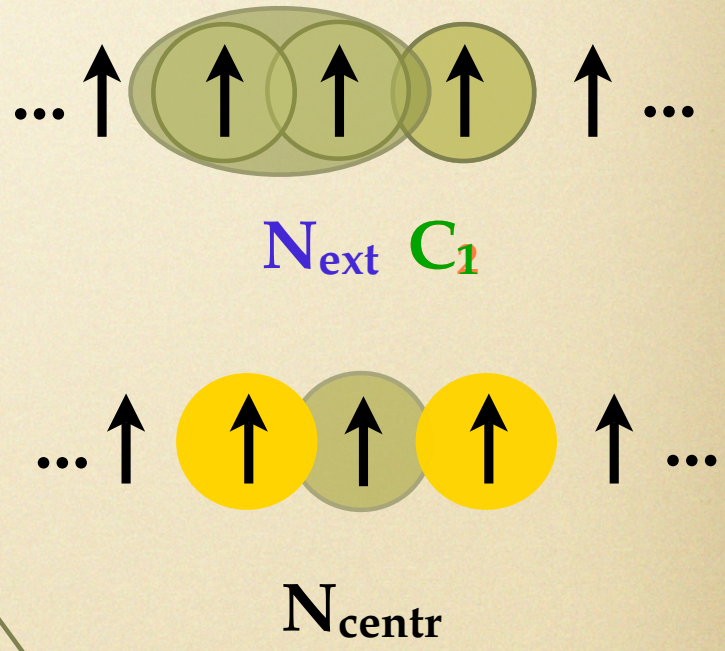


FIG. 3: Entanglement shared in a block of the three adjacent spins. We consider $\gamma = 1$. In this case only nearest neighbor and next nearest neighbor spins are entangled (hence $R = 2$) at $T = 0$ [3, 4]. The lines in the $T - h$ plane indicate the temperatures at which the corresponding type of entanglement disappears. In the marked region $T_{\mathcal{N}_{Ext}} < T < T_{\mathcal{N}_{Centr}}$ BE is present.



Bound Entanglement.

Conclusions

- Entanglement is sensitive to quantum criticality: similarities & differences with the ordinary correlators.
- Entanglement Crossover: quantum distilled of the mechanism bringing quantum effects up to finite temperatures.
- New light on traditional problems in condensed matter (Es: Kurman factorization; string-order parameter...).
- Entanglement propagation.
- Bound-entanglement in spin chains.

Amico, Fazio, Osterloh, and Vedral, quant-ph/0703044; to be published in Rev. Mod. Phys.