Entanglement in the Kondo Spin Chain

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References:
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"I feel really fortunate to have met Pasquale in 2008 for the start of what has been an amazing learning, and fruitful research experience for me! I have to say that within the mere span of two years, it already seems that I have known and worked with Pasquale for ages. Pasquale, I have always been highly impressed by your strong physical intuition, during which your take on a problem has repeatedly turned out to be true in the end. I always long for those discussions over the cigarettes, which are very pleasant indeed. I know that I am missing a great meeting but the circumstances were a bit unavoidable. I wish you a very happy 60th birthday and an enjoyable meeting and looking forward to writing several more papers together.

best wishes,
Sougato"
Contents of the Talk

Condensed Matter

2. A little bit of condensed matter
3. Kondo spin chains

Quantum Information

1. Introduction to Entanglement
4. Entanglement in the Kondo ground state
5. Generating a long range entanglement
6. Entanglement router
Pure Entangled States

Separable states:

\[ |\psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B \]
\[ |\psi\rangle_{AB} = \frac{|00\rangle + |01\rangle}{\sqrt{2}} = |0\rangle_A \otimes \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right)_B \]

Entangled states:

\[ |\psi\rangle_{AB} \neq |\alpha\rangle_A \otimes |\beta\rangle_B \]
\[ |\psi\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}} \]
\[ \rho_B = tr_A |\psi_{AB}\rangle\langle \psi_{AB}| = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| = \frac{I}{2} \]

In maximally entangled states, state of the subsystem is identity.
Non Maximal Entangled States

\[ |\psi\rangle_{AB} = \sqrt{\frac{1}{3}} |00\rangle_{AB} + \sqrt{\frac{2}{3}} |11\rangle_{AB} \]

\[ \rho_B = tr_A |\psi_{AB}\rangle\langle\psi_{AB}| = \frac{1}{3} |0\rangle\langle 0| + \frac{2}{3} |1\rangle\langle 1| \]

For entangled pure states:

- State of the subsystem is mixed.
- More mixedness in the subsystem more entanglement in the system
- Entropy of the subsystem is a unique measure of entanglement
Mixed Entangled States

Separable states:

\[ \rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B \]

There is no unique measure of entanglement

\[ E(\rho_{AB}) \]

E (Separable states) = 0

E (Maximally Entangled) = 1
Gapped Systems

\[ \Delta \iff \xi : \quad \langle S_x^i S_x^j \rangle \approx e^{-\frac{|i-j|}{\xi}} \]

The intrinsic length scale of the system impose an exponential decay
Gapless Systems

There is no length scale in the system so correlations decay algebraically
Despite the gapless nature of the Kondo system, we have a length scale in the model.
Interesting Issues of the Kondo Physic

1- Size of the cloud

2- Scaling properties in terms of the Kondo length

3- Detecting the Kondo cloud

4- Physical properties (resistively, susceptibility) in the Kondo regime
Realization of the Kondo Effect

**Semiconductor quantum dots**

**Carbon nanotubes**

**Individual molecules**
Kondo Spin Chain

\[ H = J' (J_1 \sigma_1 \cdot \sigma_2 + J_2 \sigma_1 \cdot \sigma_3) + \sum_{i=2}^{\infty} J_1 \sigma_i \cdot \sigma_{i+1} + J_2 \sigma_i \cdot \sigma_{i+2} \]

\[ \frac{J_2}{J_1} < J_2^c = 0.2412 : \quad \text{Kondo (gapless)} \]

\[ \frac{J_2}{J_1} > J_2^c : \quad \text{Dimer (gapfull)} \]

Entanglement as a Witness of the Cloud

\[ L < \xi_K : \quad E_{SA} < 1 \Rightarrow E_{SB} > 0 \]
\[ L = \xi_K : \quad E_{SA} = 1 \Rightarrow E_{SB} = 0 \]
\[ L > \xi_K : \quad E_{SA} = 1 \Rightarrow E_{SB} = 0 \]
Scaling

Kondo Phase: \( E(L, \xi_K, N) = E\left(\frac{N}{\xi_K}, \frac{L}{N}\right) \)

Dimer Phase: \( E(L, \xi, N) \neq E\left(\frac{L}{\xi_K}, \frac{N}{L}\right) \)
Scaling Properties of the Kondo regime

\[ \xi_K = e^{\alpha/\sqrt{J'}} \]

Kondo Phase:
\[ E (L, \xi_K, N) = E \left( \frac{N}{\xi_K}, \frac{L}{N} \right) \]

Dimer Phase:
\[ E (L, \xi, N) \]

\[ \frac{N}{\xi_K} = 4 \]
Local Quench

\[ H_1 = J'(J_1 \sigma_1 \cdot \sigma_2 + J_2 \sigma_1 \cdot \sigma_3) + \sum_{i=2} \sigma_i \cdot \sigma_{i+1} + J_2 \sigma_i \cdot \sigma_{i+2} \]

\[ H_2 = J'(J_1 \sigma_1 \cdot \sigma_2 + J_1 \sigma_{N-1} \cdot \sigma_N + J_2 \sigma_1 \cdot \sigma_3 + J_2 \sigma_{N-2} \cdot \sigma_N) \]
\[ + \sum_i \sigma_i \cdot \sigma_{i+1} + J_2 \sigma_i \cdot \sigma_{i+2} \]

\[ |\psi(0)\rangle = |GS_{H1}\rangle \]
\[ |\psi(t)\rangle = e^{-iH_2t} |GS_{H1}\rangle \]
\[ \rho_{1N}(t) \longrightarrow E_{1N}(t) \]
Kondo versus Dimer

(a) Energy $E$ versus time $t$ for different values of $J_2$.
(b) Energy $E$ versus $N$ for different values of $J_2$.
(c) Optimal $t_{opt}$ versus $N$ for different values of $J_2$.
(d) Optimal $J'_{opt}$ versus $N$ for different values of $J_2$. 

Symbols: $J_2 = 0$ (blue diamonds), $J_2 = 0.42$ (red triangles).
Optimal Parameter

\[ \xi_K (J') \]

\[ \xi_K = N - 2 \]

\[ \xi_K < \frac{N}{2} \]

\[ \xi_K = \frac{N}{2} \]
Optimality and Distance independence

\[ \xi_K (J'_{\text{opt}}) = N - 2 \]

Independent of length \( N \), when cloud contains \( N-2 \) spins we generate a constant Entanglement
Entanglement in Whole Phase Diagram

Entanglement drops in the dimer regime
Cloud’s Role
Entanglement in the Modified Strategy

Kondo: High value distance independent entanglement mediated by the cloud.

Dimer: Entanglement is just due to an end-end effect which decays exponentially.
Is there a way to improve the strategy?

1) Higher entanglement

2) A way to route entanglement
Two Spin Singlets

\[ J'_L \vec{\sigma}_1 \cdot \vec{\sigma}_2 \quad J'_R \vec{\sigma}_3 \cdot \vec{\sigma}_4 \]

\[ J_m \vec{\sigma}_2 \cdot \vec{\sigma}_3 \]

\[ |\psi(0)\rangle = |\psi^-\rangle \otimes |\psi^-\rangle \]

\[ |\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \]

\[ J_m = J'_L + J'_R \quad E(t) = \max \{0, \frac{1 - 3 \cos(4J_m t)}{4}\} \]
Goal

Can we find some many-body singlets (extended singlets) which play the same role?
Extended Singlet

With tuning $J'$ we can generate a proper cloud which extends up to the end of the chain.

\[ \xi_K (J'_{opt}) = N - 1 \]
Quench Dynamics

\[ \xi_L (J'_L) = N_L - 1 \]

\[ \xi_R (J'_R) = N_R - 1 \]

\[ |\psi(0)\rangle = |GS_L\rangle \otimes |GS_R\rangle \]

\[ |\psi(t)\rangle = e^{-iH_{LR}t} |\psi(0)\rangle \]

\[ \rho_{1N}(t) \quad E_{1N}(t) \]
1- Entanglement dynamics is very long lived and oscillatory
2- maximal entanglement attains a constant values for large chains
3- The optimal time which entanglement peaks is linear
For simplicity take a symmetric composite:

\[ \xi_L (J') = \frac{(N - 2)}{2} \]

\[ \xi_R (J') = \frac{(N - 2)}{2} \]

\[ E(t, N, J') = E(t, N, \xi) = E\left( \frac{t}{N}, \frac{N}{\xi} \right) = E\left( \frac{t}{N}, \frac{2N}{N - 2} \right) \]
Optimal Quench

\[ J'_m = J'_{L} + J'_{R} \]

\[ J_m = ? \]
Optimal $J_m$

$J_m$ saturates to $J_1$ for large $N$
Optimal Quench

\[ J_m = J'_L + J'_R \]

\[ J'_R(L) = \frac{1}{\log^2(N_{R(L)})} \]

\[ \Phi(N) \approx \log^2\left(\frac{N}{2}\right) \]
Dependence on $N$

$$\Phi(N) \approx \log^2 \left( \frac{N}{2} \right)$$
Non-Kondo Singlets (Dimer Regime)

Clouds are absent

<table>
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<th>$N$</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
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<th>28</th>
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<tr>
<td>$E_m(K)$</td>
<td>0.964</td>
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<td>$E_m(D)$</td>
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<tr>
<td>$t^*(K)$</td>
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<td>3.980</td>
<td>4.700</td>
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<td>$t^*(D)$</td>
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<td>35.01</td>
</tr>
</tbody>
</table>

K: Kondo (J2=0)        D: Dimer (J2=0.42)
Non-Symmetric Chains

\[ \xi_L(J'_L) = N_L - 1 \]

\[ \xi_R(J'_R) = N_R - 1 \]
Entanglement in Non-Symmetric Chains

Symmetric geometry gives the best output
Entanglement Router
Summary

- With using the entanglement measures, one can capture the properties of the Kondo physics.

- In the Kondo regime, Kondo cloud plays the role of the mediator to create a long range “distance independent” entanglement between individual ending spins.

- One can make an entanglement router through connecting Kondo spin chains.
Mechanism of Entanglement Generation

\[ H_1 : \quad |\psi(0)\rangle = |GS_1\rangle \]

\[ H_1 \rightarrow H_2 : \{ E_i , |E_i\rangle \} \]

\[ |\psi(t)\rangle = \sum_i e^{-iE_it} \langle E_i |GS_1 \rangle |E_i\rangle \]

Kondo Phase: *Only two* states dominantly involve in the dynamics

Dimer Phase: *Many* states involve in the dynamics
By quenching a single bond, we release some energy into the system:

$$\delta E \propto 1 - J'$$

Energy separation between $E_1$ and $E_2$: $\Delta E(J')$
Static Entanglement

Static strategy creates high amount of entanglement in perturbative regime ($J' \to 0$).

$$J' \propto \frac{\varepsilon}{\sqrt{N}}$$

Due to the vanishing gap this entanglement is highly unstable to thermal fluctuations.

$$\Delta = J'^2 = \frac{\varepsilon^2}{N}$$

$$KT < \Delta = \frac{\varepsilon^2}{N}$$
Dynamical Entanglement

\[ \rho_{th} = \frac{e^{-\beta H_1}}{Z} \]

\[ H_1 \rightarrow H_2 \quad \rho(t) = e^{-iH_2t} \rho_{th} e^{+iH_2t} \rightarrow \rho_{1N}(t) \rightarrow E_{1N}(t) \]

Thermal stability:

\[ KT_K = \frac{1}{\xi} = \frac{1}{N-2} \quad \rightarrow \quad KT < KT_K = \frac{1}{N-2} \]
Thermal Effect on the Entanglement

Dynamical strategy for creating entanglement is more resistive than the static one.